

# Existence of multi-peak solutions for a class of quasilinear problems in Orlicz-Sobolev spaces

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## Abstract

The aim of this work is to establish the existence of multi-peak solutions for the following class of quasilinear problems

$$-\operatorname{div}(\epsilon^2 \phi(\epsilon |\nabla u|) \nabla u) + V(x) \phi(|u|) u = f(u) \quad \text{in } \mathbb{R}^N,$$

where  $\epsilon$  is a positive parameter,  $N \geq 2$ ,  $V, f$  are continuous functions satisfying some technical conditions and  $\phi$  is a  $C^1$ -function.

**2000 Mathematics Subject Classification:** 35A15, 35J62, 46E30, 34B18

**Keywords:** Variational methods, Quasilinear problems, Orlicz-Sobolev spaces

## 1 Introduction

Several recent studies have focused on the nonlinear Schrödinger equation

$$i\epsilon \frac{\partial \Psi}{\partial t} = -\epsilon^2 \Delta \Psi + (V(z) + E)\Psi - f(\Psi) \quad \text{for all } z \in \mathbb{R}^N, \quad (NLS)$$

where  $N \geq 2$ ,  $\epsilon > 0$  is a positive parameter and  $V, f$  are continuous function verifying some conditions. This class of equation is one of the main objects of the quantum physics, because it appears in problems involving nonlinear optics, plasma physics and condensed matter physics.

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The knowledge of the solutions for the elliptic equation

$$\begin{cases} -\epsilon^2 \Delta u + V(z)u = f(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \quad (S)_\epsilon$$

has a great importance in the study of standing-wave solutions of  $(NLS)$ . The existence and concentration of positive solutions for general semilinear elliptic equations  $(S)_\epsilon$  for the case  $N \geq 3$  have been extensively studied, see for example, Floer and Weinstein [16], Oh [29], Rabinowitz [31], Wang [34], Cingolani and Lazzo [11], Ambrosetti, Badiale and Cingolani [6], Floer and Weinstein [17], Gui [22], del Pino and Felmer [12] and their references.

In the above mentioned papers, the existence, multiplicity and concentration of positive solutions have been obtained in connection with the geometry of the function  $V$ . In [31], by a mountain pass argument, Rabinowitz proves the existence of positive solutions of  $(S)_\epsilon$  for  $\epsilon > 0$  small and

$$\liminf_{|z| \rightarrow \infty} V(z) > \inf_{z \in \mathbb{R}^N} V(z) = V_0 > 0. \quad (R)$$

Later Wang [34] showed that these solutions concentrate at global minimum points of  $V$  as  $\epsilon$  tends to 0. In [12], del Pino and Felmer have found solutions which concentrate around local minimum of  $V$  by introducing a penalization method. More precisely, they assume that there is an open and bounded set  $\Lambda \subset \mathbb{R}^N$  such that

$$0 < V_0 \leq \inf_{z \in \Lambda} V(z) < \min_{z \in \partial \Lambda} V(z).$$

The existence of multi-peak solution has been considered in some papers. In [22], Gui has showed the existence of a  $\kappa$ -peak solution  $u_\epsilon$  for the problem  $(S)_\epsilon$  under the assumptions that  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  is a continuous function verifying

$$V(z) \geq V_0 > 0 \text{ for all } z \in \mathbb{R}^N \quad (V_0)$$

and there exist  $\kappa$  disjoint bounded regions  $\Omega_1, \dots, \Omega_\kappa$  such that

$$M_i = \min_{x \in \partial \Omega_i} V(x) > \alpha_i = \inf_{x \in \Omega_i} V(x) \quad i = 1, \dots, \kappa. \quad (V_1)$$

A similar result was also obtained by del Pino and Felmer in [13] by using a different approach. In [2], Alves has generalized the results found in [22] for a class of quasilinear problems involving the  $p$ -Laplacian operator. The reader can find more information about multi-peak solutions for quasilinear problems associated with  $(S)_\epsilon$  in Giacomini and Squassina [21], Zhang and Xu [35] and their references.

After a bibliography review, we did not find any paper related to the existence of multi-peak solution for quasilinear problems involving  $N$ -functions. Motivated by this fact, we are interesting in finding multi-peak positive solutions for the following class of quasilinear problems

$$\begin{cases} -\operatorname{div}(\epsilon^2 \phi(\epsilon |\nabla u|) \nabla u) + V(x) \phi(|u|) u = f(u) \text{ in } \mathbb{R}^N, \\ u \in W^{1, \Phi}(\mathbb{R}^N), \end{cases} \quad (P_\epsilon)$$

where  $\epsilon$  is a positive parameter,  $N \geq 2$ ,  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  is a continuous function satisfying  $(V_0)$  and  $(V_1)$ , and  $\phi : [0, +\infty) \rightarrow [0, +\infty)$  is a  $C^1$ -function verifying:

$(\phi_1)$   $\phi(t), (\phi(t)t)' > 0$  for all  $t > 0$ .

$(\phi_2)$  There exist  $l, m \in (1, N)$ , such that  $l \leq m < l^* = \frac{Nl}{N-l}$  and

$$l \leq \frac{\phi(t)t^2}{\Phi(t)} \leq m \quad \forall t \neq 0, \quad \text{where } \Phi(t) = \int_0^{|t|} \phi(s) s ds.$$

$(\phi_3)$  The function  $\frac{\phi(t)}{t^{m-2}}$  is nonincreasing in  $(0, +\infty)$ .

$(\phi_4)$  The function  $\phi$  is monotone.

$(\phi_5)$  There exists a constant  $c > 0$  such that

$$|\phi'(t)t| \leq c\phi(t), \quad \forall t \in [0, +\infty).$$

Hereafter, we will say that  $\Phi \in \mathcal{C}_m$  if

$$(\mathcal{C}_m) \quad \Phi(t) \geq |t|^m, \quad \forall t \in \mathbb{R}.$$

Moreover, let us denote by  $\gamma$  the following real number

$$\gamma = \begin{cases} m, & \text{if } \Phi \in \mathcal{C}_m, \\ l, & \text{if } \Phi \notin \mathcal{C}_m. \end{cases}$$

Related to the function  $f$ , we assume that it is a  $C^1$ -function satisfying

$(f_1)$  There are functions  $r, b : [0, +\infty) \rightarrow [0, +\infty)$  such that

$$\limsup_{|t| \rightarrow 0} \frac{f'(t)}{(r(|t|)|t|)'} = 0 \quad \text{and} \quad \limsup_{|t| \rightarrow +\infty} \frac{|f'(t)|}{(b(|t|)|t|)'} < +\infty.$$

(f<sub>2</sub>) There exists  $\theta > m$  such that

$$0 < \theta F(t) \leq f(t)t \quad \forall t > 0, \quad \text{where} \quad F(t) = \int_0^t f(s)ds.$$

(f<sub>3</sub>) The function  $\frac{f(t)}{t^{m-1}}$  is increasing for  $t > 0$ .

Here, the functions  $r$  and  $b$  are  $C^1$ -function satisfying the following conditions:

(r<sub>1</sub>)  $r$  is increasing.

(r<sub>2</sub>) There exists a constant  $\bar{c} > 0$  such that

$$|r'(t)t| \leq \bar{c}r(t), \quad \forall t \geq 0.$$

(r<sub>3</sub>) There exist positive constants  $r_1$  and  $r_2$  such that

$$r_1 \leq \frac{r(t)t^2}{R(t)} \leq r_2, \quad \forall t \neq 0, \quad \text{where} \quad R(t) = \int_0^{|t|} r(s)ds.$$

(r<sub>4</sub>) The function  $R$  satisfies

$$\limsup_{t \rightarrow 0} \frac{R(t)}{\Phi(t)} < +\infty \quad \text{and} \quad \limsup_{|t| \rightarrow +\infty} \frac{R(t)}{\Phi_*(t)} = 0.$$

(b<sub>1</sub>)  $b$  is increasing.

(b<sub>2</sub>) There exists a constant  $\tilde{c} > 0$  such that

$$|b'(t)t| \leq \tilde{c}b(t), \quad \forall t \geq 0.$$

(b<sub>3</sub>) There exist positive constants  $b_1, b_2 \in (1, \gamma^*)$  such that

$$b_1 \leq \frac{b(t)t^2}{B(t)} \leq b_2 \quad \forall t \neq 0, \quad \text{where} \quad B(t) = \int_0^{|t|} b(s)ds \quad \text{and} \quad \gamma^* = \frac{N\gamma}{N-\gamma}.$$

(b<sub>4</sub>) The function  $B$  satisfies

$$\limsup_{t \rightarrow 0} \frac{B(t)}{\Phi(t)} < +\infty \quad \text{and} \quad \limsup_{|t| \rightarrow +\infty} \frac{B(t)}{\Phi_*(t)} = 0,$$

where  $\Phi_*$  is the Sobolev conjugate function, which is defined as being the inverse function of

$$G_\Phi(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{1+\frac{1}{N}}} ds.$$

Using the change variable  $v(x) = u(x/\epsilon)$ , it is easy to see that the problem  $(P_\epsilon)$  is equivalent to the following problem

$$\begin{cases} -\Delta_\Phi v + V(\epsilon x)\phi(|v|)v = f(v) \text{ in } \mathbb{R}^N, \\ v \in W^{1,\Phi}(\mathbb{R}^N), \end{cases} \quad (\tilde{P}_\epsilon)$$

where the operator  $\Delta_\Phi u = \operatorname{div}(\phi(|\nabla u|)\nabla u)$ , named  $\Phi$ -Laplacian operator, is a natural extension of the  $p$ -Laplacian operator, with  $p$  being a positive constant. This operator appears in a lot of physical applications, such as

*Non-Newtonian Fluid:*  $\Phi(t) = \frac{1}{p}|t|^p$  for  $p > 1$ ,

*Plasma Physics:*  $\Phi(t) = \frac{1}{p}|t|^p + \frac{1}{q}|t|^q$  where  $1 < p < q < N$  with  $q \in (p, p^*)$ ,

*Nonlinear Elasticity:*  $\Phi(t) = (1 + t^2)^\alpha - 1$ ,  $\alpha \in (1, \frac{N}{N-2})$ ,

*Plasticity:*  $\Phi(t) = t^p \ln(1 + t)$ ,  $1 < \frac{-1+\sqrt{1+4N}}{2} < p < N-1$ ,  $N \geq 3$ ,

*Generalized Newtonian Fluid:*  $\Phi(t) = \int_0^t s^{1-\alpha}(\sinh^{-1} s)^\beta ds$ ,  $0 \leq \alpha \leq 1$  and  $\beta > 0$ .

The reader can find more details about the physical applications in [14], [17], [19] and their references. The existence of solution for  $(\tilde{P}_\epsilon)$  when  $\epsilon = 1$  in bounded and unbounded domains of  $\mathbb{R}^N$  has been established in some paper, see for example [3], [7], [8], [9], [18], [24], [25], [26], [27], [32], [33] and references therein. However, associated with the existence, multiplicity and concentration of solution for a  $\Phi$ -Laplacian equation, the authors know only the papers [4] and [5].

Now, we are ready to state our main result.

**Theorem 1.1** *Suppose that  $(\phi_1)$ -( $\phi_3$ ),  $(b_1)$ -( $b_3$ ),  $(f_1)$ -( $f_3$ ),  $(V_0)$  and  $(V_1)$  hold. Then, for each  $\Gamma \subset \{1, \dots, \kappa\}$ , there exist  $\epsilon^* > 0$  such that, for that, for  $\epsilon \in (0, \epsilon^*]$ ,  $(P_\epsilon)$  has a family  $\{u_\epsilon\}$  of positive solutions verifying the following property for  $\epsilon$  small enough:*

*There exists  $\delta > 0$  such that*

$$\sup_{x \in \mathbb{R}^N} u_\epsilon(x) \geq \delta.$$

*There exists  $P_{\epsilon,i} \in \Omega_i$  for all  $i \in \Gamma$  such that, for each  $\eta > 0$ , there exists  $\rho > 0$  verifying*

$$\sup_{x \in B_{\epsilon\rho}(P_{\epsilon,i})} u_\epsilon(x) \geq \delta \text{ for all } i \in \Gamma$$

and

$$\sup_{x \in \mathbb{R}^N \setminus \bigcup_{i \in \Gamma} B_{\epsilon \rho}(P_{\epsilon, i})} u_{\epsilon}(x) < \eta.$$

In the above theorem, if  $\Gamma$  has  $\iota$  elements, we say that  $u_{\epsilon}$  is a  $\iota$ -peak solution. From now on, we will work with  $(\tilde{P}_{\epsilon})$  to get multi-peak solutions of  $(P_{\epsilon})$ .

The proof of our main theorem will make by using variational methods and adpating some arguments found in [2] and [22]. However, we would like point out that some estimates in our paper are totally different from those used in [2, 22], because some properties and estimates that occur for the  $p$ -Laplacian do not hold for a  $\Phi$ -Laplacian equation. Here, we overcome these difficulties by showing a new version of Lions' Lemma for Orlicz-Sobolev Spaces and also a new property involving the Orlicz-Sobolev, these two results can be seen in Section 5. Moreover, in [2] was used the interaction Moser techniques, which does not work well in our case. Hence, it was necessary to change the arguments and we have used some ideas found in [20] and [23].

Before concluding this section, we would like to say that the reader can find a brief review about Orlicz-Sobolev spaces in [4], [5] and [18]. However, for a more detailed study, we cite the books [1], [28] and [30].

**Notation:** In this paper, we use the following notations:

- If  $A$  is a N-function, we denote by  $\tilde{A}$  and  $A_*$  its complementary and conjugate functions respectively.
- If  $A$  is a N-function, we denote by  $L^A(\mathbb{R}^N)$  and  $W^{1,A}(\mathbb{R}^N)$  the Orlicz and Orlicz-Sobolev spaces respectively. Moreover, we denote by  $\| \cdot \|_A$  and  $\| \cdot \|_{1,A}$  their usual norms given by

$$\|u\|_A := \inf \left\{ \lambda > 0; \int_{\mathbb{R}^N} A\left(\frac{|u|}{\lambda}\right) dx \leq 1 \right\}$$

and

$$\|u\|_{1,A} = \|\nabla u\|_A + \|u\|_A.$$

- We say that a N-function  $A$  verifies the  $\Delta_2$ -condition, denote by  $A \in \Delta_2$ , if there is a  $K > 0$  such that

$$A(2t) \leq K A(t), \quad \forall t \geq 0.$$

If  $A, \tilde{A}$  are N-functions verifying the  $\Delta_2$ -condition, then  $L^A(\mathbb{R}^N)$  and  $W^{1,A}(\mathbb{R}^N)$  are reflexive and separable. As  $(\phi_1)$ -( $\phi_2$ ) imply that  $\Phi$  and  $\tilde{\Phi}$  satisfy the  $\Delta_2$ -condition, we have  $L^\Phi(\mathbb{R}^N)$  and  $W^{1,\Phi}(\mathbb{R}^N)$  are reflexive and separable spaces.

- $C$  denotes (possible different) any positive constant, whose value is not relevant.

## 2 Penalization Method

In present section our main goal is to prove the existence of solution for an auxiliary problem by adapting some ideas explored in [2] and [22].

Since we intend to find positive solutions, we will assume that

$$f(t) = 0, \quad \forall t \in (-\infty, 0]. \quad (2.1)$$

In what follows, let us denote by  $I_\epsilon : X_\epsilon \rightarrow \mathbb{R}$  the energy functional given by

$$I_\epsilon(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|) dx + \int_{\mathbb{R}^N} V(\epsilon x) \Phi(|u|) dx - \int_{\mathbb{R}^N} F(u) dx,$$

where  $X_\epsilon$  denotes the subspace of  $W^{1,\Phi}(\mathbb{R}^N)$  given by

$$X_\epsilon = \left\{ u \in W^{1,\Phi}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x) \Phi(|u|) dx < +\infty \right\}$$

endowed with the norm

$$\|u\|_\epsilon = \|\nabla u\|_\Phi + \|u\|_{\Phi, V_\epsilon}$$

where

$$\|\nabla u\|_\Phi := \inf \left\{ \lambda > 0; \int_{\mathbb{R}^N} \Phi\left(\frac{|\nabla u|}{\lambda}\right) dx \leq 1 \right\}$$

and

$$\|u\|_{\Phi, V_\epsilon} := \inf \left\{ \lambda > 0; \int_{\mathbb{R}^N} V(\epsilon x) \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1 \right\}.$$

As  $\Phi$  and  $\tilde{\Phi}$  verify  $\Delta_2$ -condition, the space  $X_\epsilon$  is reflexive and separable. Moreover, from  $(V_0)$ , it follows that the embeddings

$$X_\epsilon \hookrightarrow L^\Phi(\mathbb{R}^N) \quad \text{and} \quad X_\epsilon \hookrightarrow L^B(\mathbb{R}^N) \quad (2.2)$$

are continuous. From the above embeddings, a direct computation yields  $I_\epsilon \in C^1(X_\epsilon, \mathbb{R})$  with

$$I'_\epsilon(u)v = \int_{\mathbb{R}^N} \phi(|\nabla u|) \nabla u \nabla v \, dx + \int_{\mathbb{R}^N} V(\epsilon x) \phi(|u|) uv \, dx - \int_{\mathbb{R}^N} f(u)v \, dx$$

for all  $u, v \in X_\epsilon$ . Thereby,  $u \in X_\epsilon$  is a weak solution of  $(\tilde{P}_\epsilon)$  if, and only if,  $u$  is a critical point of  $I_\epsilon$ . Furthermore, by (2.1), the critical points of  $I_\epsilon$  are nonnegative.

Let  $\theta$  be the number given in  $(f_3)$  and  $a, \xi > 0$  satisfying

$$\xi > \frac{(\theta - l)}{(\theta - m)} \frac{m}{l} \quad \text{and} \quad \frac{f(a)}{\phi(a)a} = \frac{V_0}{\xi}.$$

Using the above numbers, let us define the function

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s \leq a, \\ \frac{V_0}{\xi} \phi(s)s & \text{if } s > a. \end{cases}$$

Finally, fixed  $\Gamma \subset \{1, \dots, \kappa\}$ , we consider the function

$$g(x, s) = \chi_\Omega(x)f(s) + (1 - \chi_\Omega(x))\tilde{f}(s),$$

where  $\chi_\Omega$  is the characteristic function related to the set

$$\Omega = \bigcup_{i \in \Gamma} \Omega_i.$$

From definition of  $g$ , it follows that  $g$  is a Carathéodory function verifying

$$g(x, s) = 0, \quad \forall (x, s) \in \mathbb{R}^N \times (-\infty, 0] \quad (2.3)$$

and

$$g(x, s) \leq f(s), \quad \forall (x, s) \in \mathbb{R}^N \times \mathbb{R}. \quad (2.4)$$

Moreover, the following conditions also hold:

$$(g_1) \quad 0 \leq \theta G(x, s) = \theta \int_0^s g(x, t) dt \leq g(x, s)s, \quad \forall (x, s) \in \Omega \times (0, +\infty).$$

$$(g_2) \quad 0 < lG(x, s) \leq g(x, s)s \leq \frac{V_0}{\xi} \phi(s)s^2, \quad \forall (x, s) \in \Omega^c \times (0, +\infty).$$



Using the function  $g$ , we set the auxiliary problem

$$\begin{cases} -\Delta_{\Phi} u + V(\epsilon x)\phi(|u|)u &= g(\epsilon x, u) \text{ in } \mathbb{R}^N, \\ u &\in W^{1,\Phi}(\mathbb{R}^N). \end{cases} \quad (A_{\epsilon})$$

Associated with  $(A_{\epsilon})$ , we have the functional  $J_{\epsilon} : X_{\epsilon} \rightarrow \mathbb{R}$  given by

$$J_{\epsilon}(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|) dx + \int_{\mathbb{R}^N} V(\epsilon x)\Phi(|u|) dx - \int_{\mathbb{R}^N} G(\epsilon x, u) dx,$$

which belongs to  $C^1(X_{\epsilon}, \mathbb{R})$  with

$$J'_{\epsilon}(u)v = \int_{\mathbb{R}^N} \phi(|\nabla u|)\nabla u \nabla v dx + \int_{\mathbb{R}^N} V(\epsilon x)\phi(|u|)uv dx - \int_{\mathbb{R}^N} g(\epsilon x, u)v dx,$$

for all  $u, v \in X_{\epsilon}$ . Therefore, critical points of  $J_{\epsilon}$  are nonnegative weak solutions of  $(A_{\epsilon})$ .

Here, we would like to point out that if  $u_{\epsilon}$  is a positive solution of  $(A_{\epsilon})$  with  $u_{\epsilon}(x) \leq a$  for every  $x \in \mathbb{R}^N \setminus \Omega_{\epsilon}$  with  $\Omega_{\epsilon} = \Omega/\epsilon$ , then  $u_{\epsilon}$  is also a positive solution of  $(\tilde{P}_{\epsilon})$ .

## 2.1 The behavior of the $(PS)_c^*$ sequences

In what follows, we say that  $(u_n)$  is a  $(PS)_c^*$  sequence when

$$(u_n) \subset X_{\epsilon_n}, \quad J_{\epsilon_n}(u_n) \rightarrow c, \quad \|J'_{\epsilon_n}(u_n)\|_{\epsilon_n}^* \rightarrow 0 \quad \text{and} \quad \epsilon_n \rightarrow 0.$$

The main result of this section is as follows:

**Proposition 2.1** *Let  $(u_n)$  be a  $(PS)_c^*$  sequence. Then, there exist a subsequence of  $(u_n)$ , still denoted by itself, a nonnegative integer  $p$ , sequences of points  $(y_{n,j}) \subset \mathbb{R}^N$  with  $j = 1, \dots, p$  such that*

$$\epsilon_n y_{n,j} \rightarrow x_j \in \overline{\Omega} \quad \text{and} \quad |y_{n,j} - y_{n,i}| \rightarrow +\infty \quad \text{as } n \rightarrow +\infty$$

and

$$\left\| u_n(\cdot) - \sum_{j=1}^p u_{0,j}(\cdot - y_{n,j}) \varphi_{\epsilon_n}(\cdot - y_{n,j}) \right\|_{\epsilon_n} \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

where  $\varphi_{\epsilon}(x) = \varphi(x/(-\ln \epsilon))$  for  $0 < \epsilon < 1$ , and  $\varphi$  is a cut-off function which  $\varphi(z) = 1$  for  $|z| \leq 1$ ,  $\varphi(z) = 0$  for  $|z| \geq 2$  and  $|\nabla \varphi| \leq 2$ . The function  $u_{0,j} \neq 0$  is a nonnegative solution for

$$-\Delta_{\Phi} u + V_j \phi(|u|)u = g_{0,j}(x, u) \quad \text{in } \mathbb{R}^N, \quad (P^j)$$

where  $V_j = V(x_j) \geq V_0 > 0$  and  $g_{0,j}(x, u) = \lim_{n \rightarrow \infty} g(\epsilon_n x + \epsilon_n y_{n,j}, u)$ .  
Moreover, we have  $c \geq 0$  and

$$c = \sum_{j=1}^p J_{0,j}(u_{0,j}),$$

where  $J_{0,j} : W^{1,\Phi}(\mathbb{R}^N) \rightarrow \mathbb{R}$  denotes the functional given by

$$J_{0,j}(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|) dx + V_j \int_{\mathbb{R}^N} \Phi(|u|) dx - \int_{\mathbb{R}^N} G_{0,j}(x, u) dx$$

with  $G_{0,j}(x, t) = \int_0^t g_{0,j}(x, s) ds$ .

**Proof.** Let  $(u_n)$  be a  $(PS)_c^*$  sequence. Arguing as [4], there exists  $M > 0$  independent of  $n$  such that

$$\|u_n\|_{\epsilon_n} \leq M \quad \forall n \in \mathbb{N},$$

showing that  $(u_n)$  is a bounded sequence in  $W^{1,\Phi}(\mathbb{R}^N)$ . Since

$$c + o_n(1) = J_{\epsilon_n}(u_n) - \frac{1}{\theta} J'_{\epsilon_n}(u_n) u_n$$

and  $(\phi_2)$  combined with  $(g_1)$ -( $g_2$ ) give

$$J_{\epsilon_n}(u_n) - \frac{1}{\theta} J'_{\epsilon_n}(u_n) u_n \geq C \left( \int_{\mathbb{R}^N} \Phi(|\nabla u_n|) dx + \int_{\mathbb{R}^N} V(\epsilon_n x) \Phi(|u_n|) dx \right), \quad (2.5)$$

with  $C = \left[ \left(1 - \frac{m}{\theta}\right) - \left(1 - \frac{l}{\theta}\right) \frac{m}{kl} \right] > 0$ , we deduce that  $c \geq 0$ . Thereby, if  $c = 0$ , (2.5) ensures that

$$\int_{\mathbb{R}^N} \Phi(|\nabla u_n|) dx \rightarrow 0 \quad \text{and} \quad \int_{\mathbb{R}^N} V(\epsilon_n x) \Phi(|u_n|) dx \rightarrow 0,$$

leading to  $\|u_n\|_{\epsilon_n} \rightarrow 0$ . In the sequel, we will consider only the case  $c > 0$ .

We claim that there exist positive constants  $\rho, a$ , a subsequence of  $(u_n)$ , still denoted by itself, and a sequence  $(y_{n,1}) \subset \mathbb{R}^N$  such that

$$\int_{B_\rho(y_{n,1})} \Phi(|u_n(x)|) dx \geq a > 0, \quad \forall n \in \mathbb{N}. \quad (2.6)$$

Otherwise, since  $(u_n)$  is bounded in  $W^{1,\Phi}(\mathbb{R}^N)$ , a Lions-type result for Orlicz-Sobolev spaces found in [3, Theorem 1.3] gives  $u_n \rightarrow 0$  in  $L^B(\mathbb{R}^N)$ , that is,

$$\int_{\mathbb{R}^N} B(|u_n|)dx \rightarrow 0.$$

Now, the definition of  $J'_{\epsilon_n}(u_n)u_n$  together with  $(f_1)$ ,  $(\phi_2)$  and  $(b_2)$  yields

$$c_1 \int_{\mathbb{R}^N} B(|u_n|)dx + J'_{\epsilon_n}(u_n)u_n \geq l \left(1 - \frac{1}{k}\right) \left( \int_{\mathbb{R}^N} \Phi(|\nabla u_n|)dx + \int_{\mathbb{R}^N} V(\epsilon_n x) \Phi(|u_n|)dx \right),$$

showing that  $\|u_n\|_{\epsilon_n} \rightarrow 0$ . Then  $c = 0$ , which is a contradiction. Therefore (2.6) holds.

Now, setting  $w_{n,1}(x) = u_n(x + y_{n,1})$ , we see that  $(w_{n,1})$  is a bounded sequence in  $W^{1,\Phi}(\mathbb{R}^N)$ . Thus, there exist  $u_{0,1} \in W^{1,\Phi}(\mathbb{R}^N)$  and a subsequence of  $(w_{n,1})$ , still denoted by itself, such that

$$w_{n,1} \rightharpoonup u_{0,1} \quad \text{in } W^{1,\Phi}(\mathbb{R}^N).$$

The above limit and (2.6) combine to give  $u_{0,1} \neq 0$ .

Hereafter, we will show that  $u_{0,1}$  is the solution of  $(P^1)$ . For this purpose, it is crucial to show the following claim:

**Claim 2.1** *The sequence  $(\epsilon_n y_{n,1})$  is bounded. Moreover, there exists  $x_1 \in \overline{\Omega}$  such that, up to a subsequence,  $\epsilon_n y_{n,1} \rightarrow x_1$ .*

In fact, suppose by contradiction that  $(\epsilon_n y_{n,1})$  is an unbounded sequence. Then, without loss of generality, we can suppose that  $|\epsilon_n y_{n,1}| \rightarrow +\infty$ . By using the limit  $\lim_{\epsilon \rightarrow 0} \epsilon \ln \epsilon = 0$ , it is easy to check that for  $n$  large enough

$$\epsilon_n y_{n,1} + \epsilon_n x \in \mathbb{R}^N \setminus \Omega \quad \text{for } |x| < 2|\ln \epsilon_n|.$$

Once  $(u_n)$  is  $(PS)_c^*$ , setting  $v_n(x) = u_n(x) \varphi_{\epsilon_n}(x - y_{n,1})$ , we have  $(\|v_n\|_{\epsilon_n})$  is bounded in  $\mathbb{R}$  and  $J'_{\epsilon_n}(u_n)v_n = o_n(1)$ . On the other hand, a direct computation gives

$$J'_{\epsilon_n}(u_n)v_n \geq \left(l - \frac{m}{k}\right) \int_{\mathbb{R}^N} \left( \Phi(|\nabla u_{0,1}|) + V_0 \Phi(|u_{0,1}|) \right) dx + o_n(1),$$

and so,

$$\int_{\mathbb{R}^N} \left( \Phi(|\nabla u_{0,1}|) + V_0 \Phi(|u_{0,1}|) \right) dx = 0,$$

implying that  $u_{0,1} = 0$ , which is absurd. Thereby,  $(\epsilon_n y_{n,1})$  is a bounded sequence. From this, there exists  $x_1 \in \mathbb{R}^N$  such that for some subsequence,  $\epsilon_n y_{n,1} \rightarrow x_1$ . The same type of argument works to prove that  $x_1 \in \overline{\Omega}$ , which proves the Claim 2.1.

The same arguments explored in [3, Lemma 4.3] work to show that there exists a subsequence of  $(w_{n,1})$ , still denote by itself, such that

$$w_{n,1}(x) \rightarrow u_{0,1}(x) \text{ and } \nabla w_{n,1}(x) \rightarrow \nabla u_{0,1}(x) \text{ a. e. in } \mathbb{R}^N. \quad (2.7)$$

The Claim 2.1 combined with the limit above permit to conclude that  $u_{0,1}$  is solution of  $(P^1)$ .

Next, we consider  $u_n^1(x) = u_n(x) - (u_{0,1}\varphi_{\epsilon_n})(x - y_{n,1})$ . We will show that  $(u_n^1)$  is a  $(PS)_{c-J_{0,1}(u_{0,1})}^*$  sequence, that is,

$$J_{\epsilon_n}(u_n^1) \rightarrow c - J_{0,1}(u_{0,1}) \text{ and } \|J'_{\epsilon_n}(u_n^1)\|_{\epsilon_n}^* \rightarrow 0.$$

Firstly, we prove that  $J_{\epsilon_n}(u_n^1) \rightarrow c - J_{0,1}(u_{0,1})$ . For this end, from a result due to Brezis and Lieb [10], we derive that

$$J_{\epsilon_n}(u_n^1) - J_{\epsilon_n}(u_n) + J_{\epsilon_n}((u_{0,1}\varphi_{\epsilon_n})(x - y_{n,1})) = I_n + o_n(1), \quad (2.8)$$

where

$$I_n = \int_{\mathbb{R}^N} \left[ G(\epsilon_n x, u_n) - G(\epsilon_n x, u_n^1) - G(\epsilon_n x, (u_{0,1}\varphi_{\epsilon_n})(x - y_{n,1})) \right] dx.$$

Arguing as in [2, Proposition 2.4], given  $\eta > 0$ , there exists  $\rho > 0$  and  $n_0 \in \mathbb{N}$  such that for  $n \geq n_0$

$$I_n \leq \eta + \int_{|x| \geq \rho} \left| G(\epsilon_n x + \epsilon_n y_{n,1}, w_{n,1}) - G(\epsilon_n x + \epsilon_n y_{n,1}, w_{n,1} - u_{0,1}\varphi_{\epsilon_n}) \right| dx.$$

Moreover, increasing  $\rho$  if necessary, the conditions  $(\phi_1)$ - $(\phi_5)$ ,  $(r_1)$ - $(r_3)$ ,  $(b_1)$ - $(b_3)$ ,  $(f_1)$ ,  $(f_3)$  and (2.4) combine to give

$$\int_{|x| \geq \rho} \left| G(\epsilon_n x + \epsilon_n y_{n,1}, w_{n,1}) - G(\epsilon_n x + \epsilon_n y_{n,1}, w_{n,1} - u_{0,1}\varphi_{\epsilon_n}) \right| dx \leq \eta + o_n(1).$$

On the other hand, a direct calculus given us

$$J_{\epsilon_n}((u_{0,1}\varphi_{\epsilon_n})(\cdot - y_{n,1})) \rightarrow J_{0,1}(u_{0,1}). \quad (2.9)$$

The last inequalities together with (2.8) and (2.9) leads to  $J_{\epsilon_n}(u_n^1) \rightarrow c - J_{0,1}(u_{0,1})$ . A similar argument can be used to show that  $\|J'_{\epsilon_n}(u_n^1)\|_{\epsilon_n}^* \rightarrow 0$ .

As  $(u_n^1)$  is a  $(PS)_{c-J_{0,1}(u_{0,1})}^*$ , we can repeat the previous arguments to find a sequence  $(y_{n,2}) \subset \mathbb{R}^N$  verifying

$$\int_{B_\rho(y_{n,2})} \Phi(|u_n^1(x)|) dx \geq a_1 > 0. \quad (2.10)$$

We observe that the sequence  $(y_{n,2})$  can be chosen so that

$$|y_{n,2} - y_{n,1}| \rightarrow +\infty. \quad (2.11)$$

Indeed, to see why, we assume that  $(|y_{n,2} - y_{n,1}|)$  is bounded in  $\mathbb{R}$ . Thus, by (2.10), there exists  $\rho_1 > 0$  such that

$$\int_{B_\rho(y_{n,2})} \Phi(|u_n^1(x)|) dx \leq \int_{B_{\rho_1}(0)} \Phi(|w_n(x) - (u_{0,1}\varphi_{\epsilon_n})(x)|) dx$$

and so,

$$\int_{B_{\rho_1}(0)} \Phi(|w_n(x) - (u_{0,1}\varphi_{\epsilon_n})(x)|) dx \geq a_1, \quad \forall n \in \mathbb{N},$$

which is absurd, because  $w_n - u_{0,1}\varphi_{\epsilon_n} \rightarrow 0$  in  $L^\Phi(B_{\rho_1}(0))$ .

Next, repeating the above arguments, we also have that  $(w_{n,2})$  given by  $w_{n,2}(x) = u_n^1(x + y_{n,2})$  is bounded in  $W^{1,\Phi}(\mathbb{R}^N)$ , and so, there exists a solution  $u_{0,2} \in W^{1,\Phi}(\mathbb{R}^N)$  of  $(P^2)$  such that

$$w_{n,2}(x) \rightarrow u_{0,2}(x) \text{ and } \nabla w_{n,2}(x) \rightarrow \nabla u_{0,2}(x) \text{ a. e. in } \mathbb{R}^N.$$

Setting  $u_n^2(x) = u_n^1(x) - (u_{0,2}\varphi_{\epsilon_n})(x - y_{n,2})$  and arguing as above, it follows that

$$J_{\epsilon_n}(u_n^2) \rightarrow c - J_{0,1}(u_{0,1}) - J_{0,2}(u_{0,2}) \quad \text{and} \quad \|J'_{\epsilon_n}(u_n^2)\|_{\epsilon_n}^* \rightarrow 0,$$

showing that  $(u_n^2)$  is a  $(PS)_{c-J_{0,1}(u_{0,1})-J_{0,2}(u_{0,2})}^*$  sequence. Continuing with this argument, we find a sequence  $(u_n^s)$  given by

$$u_n^s(x) = u_n^{s-1}(x) - (u_{0,s}\varphi_{\epsilon_n})(x - y_{n,s})$$

with

$$J_{\epsilon_n}(u_n^s) \rightarrow c - \sum_{i=1}^s J_{0,i}(u_{0,i}) \quad \text{and} \quad \|J'_{\epsilon_n}(u_n^s)\|_{\epsilon_n}^* \rightarrow 0.$$

Finally, as in [22, Proposition 2.2], there exists  $p \in \mathbb{N}$  such that

$$J_{\epsilon_n}(u_n^p) \rightarrow 0 \quad \text{and} \quad \|J'_{\epsilon_n}(u_n^p)\|_{\epsilon_n}^* \rightarrow 0.$$

This implies that

$$\|u_n^p\|_{\epsilon_n} \rightarrow 0 \quad \text{and} \quad c = \sum_{i=1}^p J_{0,i}(u_{0,i}),$$

finishing the proof. ■

### 3 Existence of a special solution for $(\tilde{P}_\epsilon)$

Our goal is looking for a special critical point of  $J_\epsilon$  for  $\epsilon$  small enough, which will help us to prove the existence of multi-peak solutions for  $(P_\epsilon)$ .

In what follows, for each  $i \in \Gamma$ ,  $\tilde{\Omega}_{\epsilon,i}$  denote mutually disjoint open sets compactly containing  $\Omega_{\epsilon,i}$ . Hereafter, let us denote by  $E_i : W^{1,\Phi}(\mathbb{R}^N) \rightarrow \mathbb{R}$  and  $E_{\epsilon,i} : \tilde{X}_{\epsilon,i} \rightarrow \mathbb{R}$  the following functionals

$$E_i(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|) dx + \int_{\mathbb{R}^N} \alpha_i \Phi(|u|) dx - \int_{\mathbb{R}^N} F(u) dx$$

and

$$\tilde{E}_{\epsilon,i}(u) = \int_{\tilde{\Omega}_{\epsilon,i}} \Phi(|\nabla u|) dx + \int_{\tilde{\Omega}_{\epsilon,i}} V(\epsilon x) \Phi(|u|) dx - \int_{\tilde{\Omega}_{\epsilon,i}} G(\epsilon x, u) dx,$$

where  $\tilde{X}_{\epsilon,i}$  denotes the space of  $W^{1,\Phi}(\tilde{\Omega}_{\epsilon,i})$  endowed with the norm

$$\|u\|_{\tilde{X}_{\epsilon,i}} = \|\nabla u\|_{\Phi, \tilde{\Omega}_{\epsilon,i}} + \|u\|_{\Phi, V_\epsilon, \tilde{\Omega}_{\epsilon,i}}$$

where

$$\|\nabla u\|_{\Phi, \tilde{\Omega}_{\epsilon,i}} := \inf \left\{ \lambda > 0; \int_{\tilde{\Omega}_{\epsilon,i}} \Phi\left(\frac{|\nabla u|}{\lambda}\right) dx \leq 1 \right\}$$

and

$$\|u\|_{\Phi, V_\epsilon, \tilde{\Omega}_{\epsilon,i}} := \inf \left\{ \lambda > 0; \int_{\tilde{\Omega}_{\epsilon,i}} V(\epsilon x) \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1 \right\}.$$

The same type of arguments found in [4] and [5] guarantee the existence of functions  $w_i \in W^{1,\Phi}(\mathbb{R}^N)$  and  $w_{\epsilon,i} \in \tilde{X}_{\epsilon,i}$  with

$$E_i(w_i) = \mu_i, \tilde{E}_{\epsilon,i}(w_{\epsilon,i}) = \tilde{\mu}_{\epsilon,i} \quad \text{and} \quad E'_i(w_i) = \tilde{E}'_{\epsilon,i}(w_{\epsilon,i}) = 0,$$

where

$$\mu_i = \inf_{u \in W^{1,\Phi}(\mathbb{R}^N) \setminus \{0\}} \sup_{t \geq 0} E_i(tu) = \inf_{\alpha \in \Gamma_i} \sup_{t \in [0,1]} E_i(\alpha(t)),$$

$$\tilde{\mu}_{\epsilon,i} = \inf_{u \in \tilde{X}_{\epsilon,i} \setminus \{0\}} \sup_{t \geq 0} E_i(tu) = \inf_{\alpha \in \tilde{\Gamma}_{\epsilon,i}} \sup_{t \in [0,1]} E_i(\alpha(t)),$$

$$\Gamma_i = \{\alpha \in C([0,1], W^{1,\Phi}(\mathbb{R}^N)) : \alpha(0) = 0, E_i(\alpha(1)) < 0\}$$

and

$$\tilde{\Gamma}_{\epsilon,i} = \{\alpha \in C([0,1], \tilde{X}_{\epsilon,i}) : \alpha(0) = 0, \tilde{E}_{\epsilon,i}(\alpha(1)) < 0\}.$$

### 3.1 Some results about the minimax levels

The main goal this subsection is to show an important limit involving the numbers  $\mu_i$  and  $\tilde{\mu}_{\epsilon,i}$ .

**Lemma 3.1** *For each  $i \in \Gamma$ , there exist  $\sigma_0, \sigma_1 > 0$ , independent of  $\epsilon$ , such that*

$$\|u\|_{\tilde{X}_{\epsilon,i}} > \sigma_0 \quad \text{and} \quad \tilde{E}_{\epsilon,i}(u) > \sigma_1, \quad \forall u \in \tilde{\mathcal{N}}_{\epsilon,i}$$

where

$$\tilde{\mathcal{N}}_{\epsilon,i} = \{u \in \tilde{X}_{\epsilon,i} \setminus \{0\} : \tilde{E}'_{\epsilon,i}(u)u = 0\}.$$

**Proof.** Note that, for any  $u \in \tilde{\mathcal{N}}_{\epsilon,i}$ , the conditions  $(\phi_2)$  and  $(g_2)$  imply that

$$c_1 \left[ \int_{\tilde{\Omega}_{\epsilon,i}} \Phi(|\nabla u|) dx + \int_{\tilde{\Omega}_{\epsilon,i}} V(\epsilon x) \Phi(|u|) dx \right] \leq c_2 \int_{\tilde{\Omega}_{\epsilon,i}} \Phi_*(|u|) dx$$

for some positive constants  $c_1, c_2 > 0$ . The last inequality together with [3, Lemmas 2.3 and 2.5] leads to

$$c_1 (\xi_0(\|\nabla u\|_{\Phi, \tilde{\Omega}_{\epsilon,i}}) + \xi_0(\|u\|_{\Phi, V_{\epsilon}, \tilde{\Omega}_{\epsilon,i}})) \leq c_2 \xi_3(\|u\|_{\Phi_*, \tilde{\Omega}_{\epsilon,i}})$$

where  $\xi_0(t) = \min\{t^l, t^m\}$  and  $\xi_3(t) = \min\{t^{l^*}, t^{m^*}\}$ . Then, by Proposition 5.1 (see Appendix), there exists a positive constant  $M^*$ , independent of  $\epsilon$ , such that

$$c_1 (\xi_0(\|\nabla u\|_{\Phi, \tilde{\Omega}_{\epsilon,i}}) + \xi_0(\|u\|_{\Phi, V_{\epsilon}, \tilde{\Omega}_{\epsilon,i}})) \leq c_2 M^* \xi_3(\|u\|_{\tilde{X}_{\epsilon,i}}).$$

From this, there is  $\sigma_0 > 0$  satisfying

$$\|u\|_{\tilde{X}_{\epsilon,i}} > \sigma_0, \quad \forall u \in \tilde{\mathcal{N}}_{\epsilon,i}.$$

On the other hand, for any  $u \in \tilde{\mathcal{N}}_{\epsilon,i}$ , the conditions  $(\phi_2)$  and  $(g_1)$ -( $g_2$ ) give

$$\tilde{E}_{\epsilon,i}(u) \geq C (\xi_0(\|\nabla u\|_{\Phi, \tilde{\Omega}_{\epsilon,i}}) + \xi_0(\|u\|_{\Phi, V_{\epsilon}, \tilde{\Omega}_{\epsilon,i}}))$$

for some positive constant  $C$ . Therefore,

$$\tilde{E}_{\epsilon,i}(u) \geq \sigma_1,$$

for some  $\sigma_1 > 0$ . This proves the lemma.  $\blacksquare$

Our next result studies the behavior of the minimax level  $\tilde{\mu}_{\epsilon,i}$  when  $\epsilon$  goes to zero.

**Lemma 3.2** *For each  $i \in \Gamma$ , the following limit holds*

$$\tilde{\mu}_{\epsilon,i} \rightarrow \mu_i \quad \text{as } \epsilon \rightarrow 0.$$

**Proof.** To begin with, let us prove that

$$\tilde{\mu}_{\epsilon,i} \leq \mu_i + o(\epsilon). \quad (3.1)$$

In what follows, let  $w_i \in W^{1,\Phi}(\mathbb{R}^N)$  such that  $E_i(w_i) = \mu_i$  and  $E'_i(w_i) = 0$ . For  $\delta > 0$  enough small, we fix  $\vartheta \in C_0^\infty([0, +\infty), [0, 1])$  with  $\vartheta(s) = 1$  if  $s \in [0, \frac{\delta}{2}]$  and  $\vartheta(s) = 0$  if  $s \in [\delta, +\infty)$ . Using the function  $\vartheta$ , we define

$$w_{\epsilon,i}(x) = \vartheta(|\epsilon x - x_i|)w_i\left(\frac{\epsilon x - x_i}{\epsilon}\right),$$

where  $V(x_i) = \min_{y \in \overline{\Omega}_i} V(y)$ . As  $\text{supp}(w_{\epsilon,i}) \subset B_\delta(\frac{x_i}{\epsilon})$ , we derive that  $w_{\epsilon,i} \in \tilde{X}_{\epsilon,i}$ .

Furthermore, there exists  $t_{\epsilon,i} > 0$  such that  $\Psi_{\epsilon,i} := t_{\epsilon,i}w_{\epsilon,i} \in \tilde{N}_{\epsilon,i}$  and

$$\tilde{\mu}_{\epsilon,i} \leq \max_{t \geq 0} \tilde{E}_{\epsilon,i}(tw_{\epsilon,i}) = \tilde{E}_{\epsilon,i}(t_{\epsilon,i}w_{\epsilon,i}). \quad (3.2)$$

Using Lebesgue's Theorem, it is possible to prove that

$$\lim_{\epsilon \rightarrow 0} \tilde{E}_{\epsilon,i}(t_{\epsilon,i}w_{\epsilon,i}) = E_i(w_i) = \mu_i.$$

Consequently,

$$\limsup_{\epsilon \rightarrow 0} \tilde{\mu}_{\epsilon,i} \leq \mu_i. \quad (3.3)$$

Now, we will prove the inequality below

$$\mu_i \leq \liminf_{\epsilon \rightarrow 0} \tilde{\mu}_{\epsilon,i}. \quad (3.4)$$



Let  $\epsilon_n \in (0, +\infty)$  with  $\epsilon_n \rightarrow 0$  and  $v_{\epsilon_n, i} \in \tilde{X}_{\epsilon_n, i}$  be a solution of the following problem

$$\begin{cases} -\Delta_\Phi u + V(\epsilon_n x) \phi(|u|)u &= g(\epsilon_n x, u) \text{ in } \tilde{\Omega}_{\epsilon_n, i}, \\ \frac{\partial u}{\partial \nu} = 0, &\text{on } \partial \tilde{\Omega}_{\epsilon_n, i}. \end{cases} \quad (P_{\epsilon, i})$$

By Lemma 3.1, there exists  $\sigma_0 > 0$ , independent of  $n$ , such that

$$\|v_{\epsilon_n, i}\|_{X_{\epsilon_n, i}} \geq \sigma_0, \text{ for all } n \in \mathbb{N}.$$

Using the last inequality together with the Proposition 5.2 (see Appendix), there exist  $(y_{n, i}) \subset \mathbb{R}^N$ ,  $\varrho > 0$  and  $a > 0$  such that

$$\lim_{n \rightarrow +\infty} \int_{B_\varrho(y_{n, i}) \cap \Omega_{\epsilon_n, i}} \Phi(|v_{\epsilon_n, i}|) dx \geq a. \quad (3.5)$$

Moreover, by (3.5), increasing  $\varrho$  if necessary, we may assume that  $(y_{n, i}) \subset \Omega_{\epsilon_n, i}$  with  $\text{dist}(y_{n, i}, \partial \tilde{\Omega}_{\epsilon_n, i}) \rightarrow +\infty$ . Hence,  $\epsilon_n y_{n, i} \rightarrow \bar{x}_i \in \bar{\Omega}_{\epsilon_n, i}$  and given  $\rho > \varrho$ , we have  $B_{2\rho}(y_{n, i}) \subset \tilde{\Omega}_{\epsilon_n, i}$  for  $n$  sufficiently large. Setting

$$w_{n, i, \rho}(x) = \psi\left(\frac{|x|}{\rho}\right) v_{\epsilon_n, i}(x + y_{n, i}), \quad \forall x \in \tilde{\Omega}_{\epsilon_n, i} - y_{n, i}$$

where  $\psi \in C^\infty(\mathbb{R})$  is such that  $\psi = 1$  on  $[0, 1]$ ,  $\psi = 0$  on  $(2, +\infty)$ ,  $0 \leq \psi \leq 1$  and  $\psi' \in L^\infty(\mathbb{R})$ , we find

$$\int_{B_\varrho(0)} \Phi(|w_{n, i, \rho}|) dx = \int_{B_\varrho(y_{n, i})} \Phi(|v_{\epsilon_n, i}|) dx \geq a > 0.$$

Once  $\text{supp}(w_{n, i, \rho}) \subset B_{2\rho}(0)$ , we conclude that  $w_{n, i, \rho} \in W^{1, \Phi}(\mathbb{R}^N)$ . The fact that  $v_{\epsilon_n, i}$  is a solution of  $(P_{\epsilon, i})$  together with (3.1) yields there exists  $C > 0$ , independent of  $\rho$ , such that  $\|w_{n, i, \rho}\| \leq C$ . Hence, there exists  $w_\rho^i \in W^{1, \Phi}(\mathbb{R}^N)$  such that

$$w_{n, i, \rho} \rightharpoonup w_\rho^i \text{ in } W^{1, \Phi}(\mathbb{R}^N).$$

Then,  $w_{n, i, \rho} \rightarrow w_\rho^i$  in  $L_{loc}^\Phi(\mathbb{R}^N)$  and

$$\int_{B_\varrho(0)} \Phi(|w_\rho^i|) dx \geq a > 0. \quad (3.6)$$

Since  $(\|w_\rho^i\|)$  is bounded in  $\mathbb{R}$ , there exists  $w \in W^{1, \Phi}(\mathbb{R}^N)$  such that

$$w_\rho^i \rightharpoonup w \text{ in } W^{1, \Phi}(\mathbb{R}^N).$$

Thus,  $w_\rho^i \rightarrow w^i$  in  $L_{loc}^\Phi(\mathbb{R}^N)$  and

$$\int_{B_\varrho(0)} \Phi(|w^i|) dx \geq a > 0. \quad (3.7)$$

Moreover, by a direct computation,  $w^i$  is a solution of problem  $(P^i)$ , that is,

$$\int_{\mathbb{R}^N} \left[ \phi(|\nabla w^i|) \nabla w^i \nabla \zeta dx + V(\bar{x}_i) \phi(|w^i|) w^i \zeta \right] dx = \int_{\mathbb{R}^N} g(\bar{x}_i, w^i) w^i \zeta dx,$$

for all  $\zeta \in W^{1,\Phi}(\mathbb{R}^N)$ . Fixing  $\tau > \rho$ , we know that  $B_\tau(y_{n,i}) \subset \tilde{\Omega}_{\epsilon_n,i}$  for  $n$  large enough. Hence,

$$\begin{aligned} \tilde{\mu}_{\epsilon_n,i} &= \tilde{E}_{\epsilon_n,i}(v_{\epsilon_n,i}) - \frac{1}{\theta} \tilde{E}'_{\epsilon_n,i}(v_{\epsilon_n,i}) v_{\epsilon_n,i} \\ &\geq \int_{B_\tau(0)} \left[ h(|\nabla w_{n,i,\rho}|) + V(\epsilon_n x + \epsilon_n y_{n,i}) h(|w_{n,i,\rho}|) \right] dx \\ &\quad + \int_{B_\tau(0)} \left[ \frac{1}{\theta} g(\epsilon_n x, w_{n,i,\rho}) w_{n,i,\rho} - G(\epsilon_n x, w_{n,i,\rho}) \right] dx \end{aligned}$$

where  $h(t) = \Phi(t) - \frac{1}{\theta} \phi(t) t^2$ . Applying the Fatou's lemma in  $n$ , and after taking the limit of  $\rho \rightarrow +\infty$ , we derive that

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \tilde{\mu}_{\epsilon_n,i} &\geq \int_{\mathbb{R}^N} \left[ h(|\nabla w^i|) + V(\bar{x}_i) h(|w^i|) \right] dx + \int_{\mathbb{R}^N} \left[ \frac{1}{\theta} g(\bar{x}_i, w^i) w^i - G(\bar{x}_i, w^i) \right] dx \\ &= J_{0,i}(w^i) - \frac{1}{\theta} J'_{0,i}(w^i) w^i = J_{0,i}(w^i) = J_{0,i}(w^i) = \mu_{V(\bar{x}_i)} \geq \mu_i, \end{aligned}$$

showing (3.4). By (3.1) and (3.4),

$$\tilde{\mu}_{\epsilon,i} \rightarrow \mu_i \quad \text{as } \epsilon \rightarrow 0,$$

which proves the lemma. ■

### 3.2 Critical points for $J_\epsilon$

In the sequel, we fix  $\Gamma \subset \{1, \dots, \kappa\}$  and for each  $i \in \Gamma$ , we choose  $\rho_i > 1$  such that  $E_i(\rho_i^{-1} w_i), E_i(\rho_i w_i) < \mu_i$ . Setting  $\rho = \max_{i \in \Gamma} \rho_i$ , we have

$$E_i(\rho^{-1} w_i), E_i(\rho w_i) < \mu_i \quad \text{for all } i \in \Gamma \quad (3.8)$$

and

$$\mu_i = \max_{t \in [\rho^{-2}, 1]} E_i(t\rho w_i) \text{ for all } i \in \Gamma.$$

Moreover, without loss of generality, we will consider  $\Gamma = \{1, \dots, \lambda\}$  for some  $\lambda \in \{1, \dots, \kappa\}$  and define  $\tilde{H}_\epsilon : [\rho^{-2}, 1]^\lambda \rightarrow X_\epsilon$  by

$$\tilde{H}_\epsilon(\vec{\theta})(z) = \sum_{i=1}^{\lambda} \theta_i \rho(w_i \varphi)\left(z - \frac{x_i}{\epsilon}\right) \quad (3.9)$$

for all  $\vec{\theta} = (\theta_1, \dots, \theta_\lambda) \in [\rho^{-2}, 1]^\lambda$ , where  $x_i \in \Upsilon_i = \{x \in \Omega_i : V(x_i) = \alpha_i\}$ . Moreover, we set

$$\mathcal{U}_\epsilon = \left\{ H \in C([\rho^{-2}, 1]^\lambda, X_\epsilon); H = \tilde{H}_\epsilon \text{ on } \partial([\rho^{-1}, 1]^\lambda), \right. \\ \left. H(\vec{\theta})|_{\Omega_{\epsilon,i}} \neq 0 \forall i \in \Gamma \text{ and } \forall \vec{\theta} \in [\rho^{-1}, 1]^\lambda \right\}.$$

Since  $\text{supp}\left(w_i \varphi\left(z - \frac{x_i}{\epsilon}\right)\right) \subset \Omega_{\epsilon,i}$ , it follows that  $\tilde{H}_\epsilon \in \mathcal{U}_\epsilon$ . Therefore, we can define the number

$$\mathcal{S}_\epsilon = \inf_{H \in \mathcal{U}_\epsilon} \max_{\vec{\theta} \in [\rho^{-2}, 1]^\lambda} J_\epsilon(H(\vec{\theta})).$$

**Lemma 3.3** *For  $\epsilon$  small enough, the following property holds: If  $H \in \mathcal{U}_\epsilon$ , then there exists  $\vec{\theta}_* \in [\rho^{-2}, 1]^\lambda$ , such that*

$$\tilde{E}'_{\epsilon,i}(H(\vec{\theta}_*))H(\vec{\theta}_*) = 0, \text{ for all } i \in \Gamma.$$

*In particular,  $\tilde{E}_{\epsilon,i}(H(\vec{\theta}_*)) \geq \tilde{\mu}_{\epsilon,i}$ ,  $i = 1, \dots, \lambda$ .*

**Proof.** Given  $H \in \mathcal{U}_\epsilon$ , consider  $\overline{H} : [\rho^{-2}, 1]^\lambda \rightarrow \mathbb{R}^\lambda$  such that

$$\overline{H}(\vec{\theta}) = \left( \tilde{E}'_{\epsilon,1}(H(\vec{\theta}))H(\vec{\theta}), \dots, \tilde{E}'_{\epsilon,\lambda}(H(\vec{\theta}))H(\vec{\theta}) \right), \text{ where } \vec{\theta} = (\theta_1, \dots, \theta_\lambda).$$

For  $\vec{\theta} \in \partial([\rho^{-1}, 1]^\lambda)$ , it holds

$$\overline{H}(\vec{\theta}) = \left( \tilde{E}'_{\epsilon,1}(\tilde{H}_\epsilon(\vec{\theta}))\tilde{H}_\epsilon(\vec{\theta}), \dots, \tilde{E}'_{\epsilon,\lambda}(\tilde{H}_\epsilon(\vec{\theta}))\tilde{H}_\epsilon(\vec{\theta}) \right).$$

From this, we observe that there is no  $\vec{\theta} \in \partial([\rho^{-2}, 1]^\lambda)$  with  $\overline{H}(\vec{\theta}) = 0$ . In fact, for all  $i \in \Gamma$

$$\tilde{E}'_{\epsilon,i}(\tilde{H}_\epsilon(\vec{\theta}))\tilde{H}_\epsilon(\vec{\theta}) = E'_i(\theta_i \rho w_i) \theta_i \rho w_i + o_\epsilon(1) \text{ uniformly in } \vec{\theta} \in [\rho^{-2}, 1]^\lambda.$$

Thereby, if  $\vec{\theta} \in \partial([\rho^{-2}, 1]^\lambda)$ , then  $\theta_{i_0} = 1$  or  $\theta_{i_0} = \rho^{-2}$  for some  $i_0 \in \Gamma$ . Consequently,

$$0 = \tilde{E}'_{\epsilon, i_0}(\overline{H}(\vec{\theta}))\overline{H}(\vec{\theta}) = E'_{i_0}(\rho w_{i_0})\rho w_{i_0} + o_\epsilon(1)$$

or

$$0 = \tilde{E}'_{\epsilon, i_0}(\overline{H}(\vec{\theta}))\overline{H}(\vec{\theta}) = E'_{i_0}(\rho^{-2} w_{i_0})\rho^{-2} w_{i_0} + o_\epsilon(1).$$

Therefore, if  $\tilde{E}'_{\epsilon, i_0}(\overline{H}(\vec{\theta}))\overline{H}(\vec{\theta}) = 0$ , the limit of  $\epsilon \rightarrow 0$  gives

$$E'_{i_0}(\rho w_{i_0})\rho w_{i_0} = 0 \quad \text{or} \quad E'_{i_0}(\rho^{-2} w_{i_0})\rho^{-2} w_{i_0} = 0$$

from where it follows that

$$E_{i_0}(\rho w_i) \geq \mu_i \quad \text{or} \quad E_{i_0}(\rho^{-2} w_i) \geq \mu_i,$$

Thereby, there exists  $\vec{\theta}_* \in (\delta^{-1}, 1)^\lambda$  satisfying

$$\tilde{E}'_{\epsilon, i}(H(\vec{\theta}_*))H(\vec{\theta}_*) = 0, \quad \text{for all } i \in \Gamma.$$

■

The next result establishes an important relation between  $\mathcal{S}_\epsilon$  and the levels  $\mu_i$ . In what follows, we consider  $\mathcal{D}_\Gamma = \sum_{i=1}^\lambda \mu_i$ . By using the same ideas found in [2], it is possible to prove the following results

**Proposition 3.1** *The following limit holds*

$$\lim_{\epsilon \rightarrow 0} \mathcal{S}_\epsilon = \mathcal{D}_\Gamma.$$

**Corollary 3.1** *For each  $\alpha > 0$ , there exists  $\epsilon_0 = \epsilon_0(\alpha)$  such that*

$$\sup_{\vec{\theta} \in [\delta^{-1}, 1]^\lambda} J_\epsilon(\tilde{H}_\epsilon(\vec{\theta})) \leq \mathcal{D}_\Gamma + \frac{\alpha}{2} \quad \forall \epsilon \in (0, \epsilon_0).$$

Next, we will introduce some notations. Firstly, we fix the set

$$\mathcal{Z}_{\epsilon, i} = \left\{ u \in \tilde{X}_{\epsilon, i} : \|u\|_{\tilde{X}_{\epsilon, i}} \leq \frac{\sigma_0}{2} \right\}$$

where  $\sigma_0 > 0$  is a constant such that

$$\liminf_{\epsilon \rightarrow 0} \|\tilde{H}_\epsilon(\vec{\theta})\|_{\tilde{X}_{\epsilon, i}} > \sigma_0 \quad \text{uniformly in } \vec{\theta} \in [\rho^{-2}, 1]^\lambda \text{ and } i \in \Gamma \quad (\text{See Lemma 3.1}).$$

Hence, there exist positive constants  $\tau$  and  $\epsilon^*$  such that

$$\text{dist}_{\epsilon,i}(\tilde{H}_\epsilon(\vec{\theta}), \mathcal{Z}_{\epsilon,i}) > \tau \text{ for all } \vec{\theta} \in [\delta^{-2}, 1]^\lambda, i \in \Gamma \text{ and } \epsilon \in (0, \epsilon^*),$$

where  $\text{dist}_{\epsilon,i}(A, B)$  denotes the distance between sets  $A$  and  $B$  of  $\tilde{X}_{\epsilon,i}$ . Moreover, we define

$$\Theta = \{u \in X_\epsilon : \text{dist}_{\epsilon,i}(u, \mathcal{Z}_{\epsilon,i}) \geq \tau \text{ for all } i \in \Gamma\}$$

and for any  $c, \mu > 0$  and  $0 < \delta < \frac{\tau}{2}$ , we consider the sets

$$J_\epsilon^c = \{u \in X_\epsilon : J_\epsilon(u) \leq c\} \text{ and } \mathcal{Q}_{\epsilon,\mu} = \{u \in \Theta_{2\delta} : |J_\epsilon(u) - \mathcal{S}_\epsilon| \leq \mu\},$$

where  $\Theta_s$ , for  $s > 0$ , denotes the set

$$\Theta_s = \{u \in X_\epsilon : \text{dist}(u, \Theta) \leq s\}.$$

Observe that for each  $\mu > 0$ , there exists  $\epsilon_1 = \epsilon_1(\mu) > 0$  such that the function  $U_\epsilon$  given by

$$U_\epsilon(z) = \sum_{i=1}^{\lambda} \left( w_i \varphi\left(z - \frac{x_i}{\epsilon}\right) \right)$$

verifies

$$U_\epsilon \in \Theta_s \text{ for all } \epsilon \in (0, \epsilon_1) \text{ and } J_\epsilon(U_\epsilon) = \mathcal{D}_\Gamma + o_\epsilon(1).$$

As  $\mathcal{S}_\epsilon = \mathcal{D}_\Gamma + o_\epsilon(1)$ , we have

$$J_\epsilon(U_\epsilon) = \mathcal{S}_\epsilon + o_\epsilon(1),$$

showing that  $\mathcal{Q}_{\epsilon,\mu} \neq \emptyset$ .

Next, let us consider  $M$  large enough, independent of  $\epsilon$ , satisfying

$$\|\tilde{H}_\epsilon(\vec{\theta})\|_\epsilon \leq \frac{M}{2} \text{ for all } \vec{\theta} \in [\rho^{-2}, 1]^{2\lambda}. \quad (3.10)$$

For each  $s > 0$ , we denote by  $\overline{B}_s = \{u \in X_\epsilon : \|u\|_\epsilon \leq s\}$  and define the number

$$\mu_* = \min \left\{ \frac{\mu_i}{4}, \frac{M}{4}, \frac{\delta}{4}; i \in \Gamma \right\}.$$

The result below establishes the existence of a special critical point for functional  $J_\epsilon$ , which will be used later on. However, we will omit its proof because it follows by using the same approach explored in [2].

**Proposition 3.2** *For each  $\mu \in (0, \mu_*)$ , there exists  $\epsilon_\mu > 0$  such that  $J_\epsilon$  has a critical point  $v_\epsilon \in \mathcal{Q}_{\epsilon,\mu} \cap \overline{B}_{M+1} \cap J_\epsilon^{\mathcal{D}_\Gamma}$  for all  $\epsilon \in (0, \epsilon_\mu)$ .*

## 4 The existence of multi-peak positive solutions

In this section, we will show existence of  $\lambda$ -peak solution for  $(P_\epsilon)$ . For this purpose, we need of the following technical lemma

**Lemma 4.1** *There exist  $\bar{\epsilon}, \bar{\mu}$ , such that the solution  $v_\epsilon$  obtained in Proposition 3.2 satisfies*

$$\max_{z \in \partial\Omega_\epsilon} v_\epsilon(z) < a \text{ for all } \mu \in (0, \bar{\mu}) \text{ and } \epsilon \in (0, \bar{\epsilon}).$$

**Proof.** Assume by contradiction, that there exist  $\epsilon_n, \mu_n \rightarrow 0$  such that

$$v_n := v_{\epsilon_n} \in \mathcal{Q}_{\epsilon_n, \mu_n} \text{ and } \max_{z \in \partial\Omega_{\epsilon_n}} v_n(z) \geq a \text{ for all } n \in \mathbb{N}.$$

Since  $v_n \in \mathcal{Q}_{\epsilon_n, \mu_n}$ , we know that

$$J'_{\epsilon_n}(v_n) = 0, \quad |J_{\epsilon_n}(v_n) - \mathcal{S}_{\epsilon_n}| \rightarrow 0 \text{ and } \text{dist}(v_n, \Theta) \leq 2\delta. \quad (4.1)$$

Applying the Proposition 2.1, there exist a nonnegative integer  $p$ , sequences of points  $(y_{n,i}) \subset \mathbb{R}^N$ , points  $x_i \in \bar{\Omega}$ ,  $i = 1, \dots, \lambda$  and functions  $u_{0,i}$  verifying

$$\left\| v_n(\cdot) - \sum_{i=1}^p u_{0,i}(\cdot - y_{n,i}) \varphi_{\epsilon_n}(\cdot - y_{n,i}) \right\|_{\epsilon_n} \rightarrow 0 \text{ as } n \rightarrow +\infty \quad (4.2)$$

and

$$\epsilon_n y_{n,i} \rightarrow x_i \text{ for } i = 1, \dots, p. \quad (4.3)$$

From (4.1), (4.2) and (4.3),  $p = \lambda$  and  $x_i \in \Upsilon_i$  for all  $i = 1, \dots, \lambda$ .

In what follows, we fix  $(z_n) \subset \partial\Omega_{\epsilon_n}$  such that

$$v_n(z_n) = \max_{z \in \partial\Omega_\epsilon} v_{\epsilon_n}(z)$$

and the function  $w_n(x) = v_n(x + z_n)$ . Then,

$$\left\| w_n(\cdot) - \sum_{i=1}^p u_{0,i}(\cdot + z_n - y_{n,i}) \varphi_{\epsilon_n}(\cdot + z_n - y_{n,i}) \right\|_{W^{1,\Phi}(\mathbb{R}^N)} \rightarrow 0 \text{ as } n \rightarrow +\infty$$

On the other hand, for each  $\varrho > 0$ ,

$$\left\| \sum_{i=1}^p u_{0,i}(\cdot + z_n - y_{n,i}) \varphi_{\epsilon_n}(\cdot + z_n - y_{n,i}) \right\|_{W^{1,\Phi}(B_\varrho(0))} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Consequently,

$$\|w_n\|_{W^{1,\Phi}(B_\varrho(0))} \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (4.4)$$

Notice that  $w_n$  is solution of problem

$$-\Delta_\Phi w_n + V(\epsilon_n x + \epsilon_n z_n) \phi(|w_n|) w_n = g(\epsilon_n x + \epsilon_n z_n, w_n) \text{ in } \mathbb{R}^N,$$

because  $v_n$  is a solution of  $(A_\epsilon)$ . Arguing as in [5, Lemma 3.2], there exists  $w \in C_{loc}^{1,\alpha}(\mathbb{R}^N)$  such that, up to a subsequence,

$$w_n \rightarrow w \text{ in } C_{loc}^{1,\alpha}(\mathbb{R}^N).$$

Since

$$\max_{z \in \partial\Omega_{\epsilon_n}} v_n(z) \geq a,$$

we have that  $w_n(0) \geq a$  for all  $n \in \mathbb{N}$ , and so,  $w(0) \geq a$ . Thereby, there exists  $\varrho \in (0, 1)$  such that  $w(x) \geq \frac{a}{2}$  for all  $x \in B_\varrho(0)$ . Consequently  $w \neq 0$ , which is a contradiction with (4.4), showing the lemma.  $\blacksquare$

#### 4.1 Proof of Theorem 1.1

By Lemma 4.1, there exist  $\bar{\epsilon}, \bar{\mu} > 0$ , such that the solution  $v_\epsilon \in \mathcal{Q}_{\epsilon, \mu}$  obtained in Proposition 3.2 satisfies

$$\max_{z \in \partial\Omega_\epsilon} v_\epsilon(z) < a \text{ for all } \mu \in (0, \bar{\mu}) \text{ and } \epsilon \in (0, \bar{\epsilon}).$$

Repeating the same arguments found in [5], we see that

$$v_\epsilon(x) \leq a \text{ for all } x \in \mathbb{R}^N \setminus \Omega_\epsilon.$$

Hence,  $v_\epsilon$  is a solution of  $(\tilde{P}_\epsilon)$  for all  $\epsilon \in (0, \bar{\epsilon})$ . To finish the proof, we will show that the family  $(v_\epsilon)$  is a  $\lambda$ -peak solution. To see why, we consider  $\epsilon_n \rightarrow 0$  and  $v_n = v_{\epsilon_n}$ . Observe that  $(v_n)$  is a  $(PS)_{D_\Gamma}^*$  sequence verifying

$$\text{dist}(v_n, \Theta) \leq 2\delta \text{ for all } n \in \mathbb{N}. \quad (4.5)$$

From Proposition 2.1, there exist a subsequence of  $(v_n)$ , still denoted by itself, a nonnegative integer  $p$ , sequences of points  $(y_{n,i}) \subset \mathbb{R}^N$  with  $i = 1, \dots, p$  such that

$$\epsilon_n y_{n,i} \rightarrow x_i \in \bar{\Omega} \quad \text{and} \quad |y_{n,j} - y_{n,i}| \rightarrow +\infty \quad \text{as } n \rightarrow +\infty \quad (4.6)$$

with  $i \in \{1, \dots, p\}$  and

$$\left\| v_n(\cdot) - \sum_{i=1}^p u_{0,i}(\cdot - y_{n,i}) \varphi_{\epsilon_n}(\cdot - y_{n,i}) \right\|_{\epsilon_n} \rightarrow 0 \text{ as } n \rightarrow +\infty \quad (4.7)$$

where  $\varphi_\epsilon(x) = \varphi(x/(-\ln \epsilon))$  for  $0 < \epsilon < 1$ , and  $\varphi$  is a cut-off function which  $\varphi(z) = 1$  for  $|z| \leq 1$ ,  $\varphi(z) = 0$  for  $|z| \geq 2$  and  $|\nabla \varphi| \leq 2$ . The function  $u_{0,j} \neq 0$  is a nonnegative solution for

$$-\Delta_\Phi u + V_i \phi(|u|)u = g_{0,i}(x, u) \quad \text{in } \mathbb{R}^N,$$

where  $V_i = V(x_i) \geq V_0 > 0$  and  $g_{0,i}(x, u) = \lim_{n \rightarrow \infty} g(\epsilon_n x + \epsilon_n y_{n,i}, u)$ . Furthermore,

$$\sum_{i=1}^\lambda \mu_i = \sum_{i=1}^p J_{0,i}(u_{0,i}) \quad (4.8)$$

where  $J_{0,i} : W^{1,\Phi}(\mathbb{R}^N) \rightarrow \mathbb{R}$  denotes the functional given by

$$J_{0,i}(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|) dx + V_i \int_{\mathbb{R}^N} \Phi(|u|) dx - \int_{\mathbb{R}^N} G_{0,i}(x, u) dx$$

with  $G_{0,i}(x, t) = \int_0^t g_{0,i}(x, s) ds$ . Arguing as in proof of Lemma 4.1 and using (4.5)-(4.8), we infer that  $p = \lambda$ ,  $x_i \in \overline{\Omega}_i$  and

$$\sum_{j=1}^\lambda \mu_j = \sum_{j=1}^\lambda J_{0,j}(u_{0,j}).$$

The last equality yields  $x_i \in \Upsilon_i$  and  $V(x_i) = \alpha_i$ , because if for some  $i_0 \in 1, \dots, \lambda$ , we have  $x_{i_0} \in \partial\Omega_{i_0}$ , the assumption  $(V_1)$  leads to  $V(x_{i_0}) > \alpha_{i_0}$ , and so,  $J_{0,i_0}(u_{0,i_0}) > \mu_{i_0}$ .

On the other hand, since  $J_{0,i}(u_{0,i_0}) \geq \mu_i$ , for all  $i = 1, \dots, \lambda$ , we must have

$$\sum_{i=1}^\lambda \mu_i < \sum_{i=1}^p J_{0,i}(u_{0,i}),$$

which is a contradiction. Therefore,  $V(x_i) = \alpha_i$  for  $i = 1, \dots, \lambda$  and  $u_{0,i}$  is a nontrivial solution of problem

$$-\Delta_\Phi u + \alpha_i \phi(|u|)u = f(u) \quad \text{in } \mathbb{R}^N.$$



Now, we will show that for each  $\eta > 0$ , there exists  $\rho > 0$  such that

$$\|v_n\|_{\infty, \mathbb{R}^N \setminus \cup_{j=1}^p B_\rho(y_{n,i})} \leq \eta \quad (4.9)$$

and there exists  $\delta > 0$  such that

$$\|v_n\|_{\infty, B_\rho(y_{n,j})} \geq \delta, \text{ for all } j \in \Gamma. \quad (4.10)$$

To this end, we need of the following estimate:

**Claim 4.1** *Given  $\eta > 0$ , there exist  $\rho > 0$  and  $n_0 \in \mathbb{N}$  such that*

$$\|v_n\|_{W^{1,\Phi}(\mathbb{R}^N \setminus \cup_{j=1}^p B_\rho(y_{n,i}))} \leq \eta, \quad \forall n \geq n_0. \quad (4.11)$$

In fact, for each  $j \in \Gamma$ , there exists  $\rho_j > 0$  such that

$$\|u_{0,j}\|_{W^{1,\Phi}(\mathbb{R}^N \setminus B_{\rho_j}(0))} < \eta.$$

Setting  $\rho = \max\{\rho_1, \dots, \rho_p\}$ , we have

$$\int_{\mathbb{R}^N \setminus B_\rho(0)} \Phi(|\nabla u_{0,j}|) dx, \quad \int_{\mathbb{R}^N \setminus B_\rho(0)} \Phi(|u_{0,j}|) dx < \eta \text{ for all } j \in \Gamma.$$

Notice that

$$\int_{\mathbb{R}^N \setminus B_\rho(y_{n,j})} \Phi(|\nabla(u_{0,j}(\cdot - y_{n,i})\varphi_{\epsilon_n}(\cdot - y_{n,j}))|) dx = \int_{\mathbb{R}^N \setminus B_\rho(0)} \Phi(|\nabla(u_{0,j}\varphi_{\epsilon_n})|) dx.$$

From  $\Delta_2$ -condition, we get

$$\int_{\mathbb{R}^N \setminus B_\rho(0)} \Phi(|\nabla(u_{0,j}\varphi_{\epsilon_n})|) dx \leq c_1 \int_{\mathbb{R}^N \setminus B_\rho(0)} \Phi(|\nabla u_{0,j}|) dx + c_2 \int_{\mathbb{R}^N \setminus B_\rho(0)} \Phi(|u_{0,j}|) dx.$$

Thereby, given  $\eta > 0$ , we can find  $\rho$  large enough verifying

$$\int_{\mathbb{R}^N \setminus B_\rho(y_{n,j})} \Phi(|\nabla(u_{0,j}(\cdot - y_{n,i})\varphi_{\epsilon_n}(\cdot - y_{n,j}))|) dx < \frac{\eta}{2}.$$

Similarly,

$$\int_{\mathbb{R}^N \setminus B_\rho(y_{n,j})} \Phi(|u_{0,j}(\cdot - y_{n,i})\varphi_{\epsilon_n}(\cdot - y_{n,j})|) dx < \frac{\eta}{2},$$

showing that

$$\|u_{0,j}(\cdot - y_{n,j})\varphi_{\epsilon_n}(\cdot - y_{n,j})\|_{W^{1,\Phi}(\mathbb{R}^N \setminus B_\rho(y_{n,j}))} \leq \eta. \quad (4.12)$$

Now, the claim follows from (4.7) and (4.12).

Using the above information, we are able to prove the following estimate

**Claim 4.2** *Given  $\eta > 0$ , there are  $\rho > 0$  and  $n_0 \in \mathbb{N}$  such that*

$$|v_n(z)| \leq \eta \text{ for all } z \in \mathbb{R}^N \setminus \cup_{j=1}^p B_{\rho+1}(y_{n,i}), \quad \forall n \geq n_0.$$

Indeed, fix  $R_1 \in (0, 1)$  and  $x_0 \in \mathbb{R}^N \setminus \cup_{j=1}^p B_{\rho+1}(y_{n,i})$  such that

$$B_{\frac{R_1}{2}}(x_0) \subset \mathbb{R}^N \setminus \cup_{j=1}^p B_{\rho}(y_{n,i}).$$

Next, for each  $h, \eta > 0$ , let us consider

$$\sigma_h = \frac{R_1}{2} + \frac{R_1}{2^{h+1}}, \quad \bar{\sigma}_h = \frac{\sigma_h + \sigma_{h+1}}{2} \quad \text{and} \quad K_h = \frac{\eta}{2} \left(1 - \frac{1}{2^{h+1}}\right) \quad \forall h = 0, 1, 2, \dots$$

Note that,

$$\sigma_h \downarrow \frac{R_1}{2}, \quad K_h \uparrow \frac{\eta}{2} \quad \text{and} \quad \sigma_{h+1} < \bar{\sigma}_h < \sigma_h < 1.$$

In what follows, let us consider

$$A_{n,K_h,\sigma_h} = \{x \in B_{\sigma_h}(x_0) : v_n(x) > K_h\}.$$

For each  $h = 0, 1, \dots$ , we fix

$$J_{h,n} = \int_{A_{n,K_h,\sigma_h}} ((v_n - K_h)_+)^{\gamma^*} dx \quad \text{and} \quad \xi_h(x) = \xi \left( \frac{2^{h+1}}{R_1} \left( |x - x_0| - \frac{R_1}{2} \right) \right),$$

where  $\xi \in C^1(\mathbb{R})$  satisfies

$$0 \leq \xi \leq 1, \quad \xi(t) = 1, \text{ for } t \leq \frac{1}{2} \quad \xi(t) = 0 \text{ for } t \geq \frac{3}{4} \quad \text{and} \quad |\xi'| < c.$$

Repeating the arguments explored in [4, Lemma 3.5], we can guarantee that

$$J_{h+1,n} \leq C A^h J_{h,n}^{1+\tau},$$

where  $C = C(N, \gamma, \gamma^*, R_1, \eta)$ ,  $\tau = \frac{\gamma^*}{\gamma} - 1$  and  $A = 2^\beta$  for some  $\beta$  sufficient large. We claim that there is  $n_0 \in \mathbb{N}$  such that

$$J_{0,n} \leq C^{\frac{1}{\tau}} A^{-\frac{1}{\tau^2}}, \quad \forall n \geq n_0. \quad (4.13)$$

Indeed, note that

$$\begin{aligned} J_{0,n} &= \int_{A_{n,K_0,\sigma_0}} \left(v_n - \frac{\eta}{2}\right)_+^{\gamma^*} dx \leq \int_{A_{n,\frac{\eta}{2},R_1}} (v_n)_+^{\gamma^*} dx \\ &\leq \left(\frac{\eta}{2}\right)^{l^*} \int_{\mathbb{R}^N \setminus \cup_{j=1}^p B_{\rho}(y_{n,i})} \left(\frac{2v_n}{\eta}\right)^{l^*} dx \\ &\leq c_1 \int_{\mathbb{R}^N \setminus \cup_{j=1}^p B_{\rho}(y_{n,i})} \Phi_*(|v_n|) dx, \end{aligned}$$

where  $c_1$  depends on  $\eta$ . On the other hand, by Proposition 5.1 (see Appendix), there is  $c_2 > 0$  independent of  $\rho$  such that

$$\int_{\mathbb{R}^N \setminus \cup_{j=1}^p B_\rho(y_{n,i})} \Phi_*(|v_n|) dx \leq c_2 \|v_n\|_{W^{1,\Phi}(\mathbb{R}^N \setminus \cup_{j=1}^p B_\rho(y_{n,i}))}.$$

Hence,

$$J_{0,n} \leq c_3 \|v_n\|_{W^{1,\Phi}(\mathbb{R}^N \setminus \cup_{j=1}^p B_\rho(y_{n,i}))},$$

where  $c_3 > 0$  depends on  $\eta$ . Now, using Claim 4.1, we can increase  $\rho$ , if necessary, of a way that

$$c_3 \|v_n\|_{W^{1,\Phi}(\mathbb{R}^N \setminus \cup_{j=1}^p B_\rho(y_{n,i}))} \leq C^{\frac{1}{\tau}} A^{-\frac{1}{\tau^2}},$$

showing (4.13). Thus, by [23, Lemma 4.7],

$$\lim_{h \rightarrow +\infty} J_{h,n} = 0.$$

On the other hand,

$$\lim_{h \rightarrow +\infty} J_{h,n} = \lim_{h \rightarrow +\infty} \int_{A_{n,K_h,\sigma_h}} ((v_n - K_h)_+)^{\gamma^*} dx = \int_{A_{n,\frac{\eta}{2},\frac{R_1}{2}}} ((v_n - \frac{\eta}{2})_+)^{\gamma^*} dx,$$

leading to

$$v_n(z) \leq \frac{\eta}{2}, \quad z \in B_{\frac{R_1}{2}}(x_0),$$

and so

$$|v_n(z)| \leq \frac{\eta}{2}, \quad z \in \mathbb{R}^N \setminus \cup_{j=1}^p B_{\rho+1}(y_{n,i}),$$

finishing the proof of the claim.

Hereafter, we consider the function  $w_{n,i}(x) = v_n(x + y_{n,i})$ . Note that it is a nonnegative and nontrivial solution of the problem

$$-\Delta_\Phi w_{n,i} + V(\epsilon_n x + \epsilon_n y_{n,i}) \phi(|w_{n,i}|) w_{n,i} = g(\epsilon_n x + \epsilon_n y_{n,i}, w_{n,i}) \quad \text{in } \mathbb{R}^N \quad (A_{\epsilon_n})$$

**Claim 4.3** *There exists  $\delta > 0$  such that  $\|w_{n,i}\|_\infty \geq \delta$  for  $n$  sufficient large.*

In fact, if  $\|w_{n,i}\|_\infty \rightarrow 0$ ,  $(f_1)$  combined with  $g$  gives

$$\frac{g(\epsilon_n x + \epsilon_n y_{n,i}, w_{n,i})}{\phi(|w_{n,i}|) w_{n,i}} \leq \frac{V_0}{2} \quad \forall n \geq n_0, \quad (4.14)$$

for some  $n_0 \in \mathbb{N}$ . Now, (4.14) together with  $J'_{\epsilon_n}(w_{n,i})w_{n,i} = 0$  leads to

$$\int_{\mathbb{R}^N} \phi(|\nabla w_{n,i}|) |\nabla w_{n,i}|^2 dx + \int_{\mathbb{R}^N} V(\epsilon_n x + \epsilon_n y_{n,i}) \phi(|w_{n,i}|) |w_{n,i}|^2 dx = 0 \quad \forall n \geq n_0,$$

from where it follows that  $\|w_{n,i}\|_{\epsilon_n} = 0$  for all  $n \geq n_0$ , which contradicts Lemma 3.1.

In the sequel, for  $\eta < \delta$ , the Claims 4.2 and 4.3 give

$$\|w_{n,i}\|_{\infty, B_{(\rho+1)}(0)} \geq \delta,$$

that is,

$$\|v_n\|_{\infty, B_{\rho+1}(y_{n,i})} \geq \delta, \quad \text{for all } i \in \Gamma.$$

Finally, setting  $u_n(x) = v_n\left(\frac{x}{\epsilon_n}\right)$  and  $P_{n,i} = \epsilon_n y_{n,i}$ , we get that  $u_n$  is a solution of  $(P_\epsilon)$  verifying

$$\|u_n\|_{\infty, B_{\epsilon_n(\rho+1)}(P_{n,i})} \geq \delta, \quad \text{for all } i \in \Gamma.$$

and

$$\|u_n\|_{\infty, \mathbb{R}^N \setminus \cup_{i \in \Gamma} B_{\epsilon_n(\rho+1)}(P_{n,i})} \leq \|v_n\|_{\infty, \mathbb{R}^N \setminus B_{\rho+1}(y_{n,i})} \leq \eta \quad \text{for all } n \geq n_0,$$

proving the theorem.

## 5 Appendix: New properties involving Orlicz-Sobolev spaces

In this appendix, we will prove some results which were used in the present paper. Our first result is associated with an important property involving Orlicz-Sobolev spaces, which is well known for Sobolev spaces. Here, we follow the same steps found in [15, Theorem 3.2] (or [1, Theorem 8.35]), however our proof can be applied for unbounded domains.

**Proposition 5.1** *There exists  $M^* > 0$ , which is independent of  $\epsilon$ , such that*

$$\|u\|_{\Phi_*, \Omega_{\epsilon,i}} \leq M^* \|u\|_{\tilde{X}_{\epsilon,i}} \quad \text{for all } u \in \tilde{X}_{\epsilon,i}.$$

**Proof.** In what follows, we define  $v(t) = (\Phi_*(t))^{1-\frac{1}{N}}$ . Firstly, notice that

$$\left| \frac{d}{dt} v(t) \right| \leq \frac{N-1}{N} \tilde{\Phi}^{-1}(v(t)^{\frac{N}{N-1}}) \quad \text{for all } t > 0. \quad (5.1)$$

For each  $u \in \tilde{X}_{\epsilon,i} \cap L^\infty(\Omega_{\epsilon,i})$  and  $k > 0$ , the function  $\nu := v \circ \left(\frac{|u|}{k}\right) \in W^{1,1}(\Omega_{\epsilon,j})$  and

$$\frac{\partial \nu(x)}{\partial x_j} = v' \left( \frac{|u|}{k}(x) \right) \frac{\operatorname{sgn} u(x)}{k} \frac{\partial u(x)}{\partial x_j}.$$

By [1, Theorem 4.12],, once  $\Omega_{\epsilon,j}$  verifies the uniform cone condition for all  $\epsilon > 0$ , we know that the constant associated with the embedding  $W^{1,1}(\Omega_{\epsilon,j}) \hookrightarrow L^{\frac{N}{N-1}}(\Omega_{\epsilon,j})$  does not depend on  $\epsilon$ , that is, there exists a positive constant  $C$ , which is independent of  $u$  and  $\epsilon$ , such that

$$\|\nu\|_{L^{\frac{N}{N-1}}(\Omega_{\epsilon,j})} \leq C \left( \sum_{j=1}^N \left\| \frac{\partial \nu}{\partial x_j} \right\|_{L^1(\Omega_{\epsilon,j})} + \|\nu\|_{L^1(\Omega_{\epsilon,j})} \right),$$

or equivalently,

$$\left[ \int_{\Omega_{\epsilon,i}} \Phi_* \left( \frac{|u|}{k} \right) dx \right]^{1-\frac{1}{N}} \leq \frac{C}{k} \sum_{j=1}^N \int_{\Omega_{\epsilon,i}} \left| v' \left( \frac{|u|}{k} \right) \frac{\partial u}{\partial x_j} \right| dx + C \int_{\Omega_{\epsilon,i}} \left| v \left( \frac{|u|}{k} \right) \right| dx.$$

Setting  $k = \|u\|_{\Phi_*, \Omega_{\epsilon,i}}$ , the Holder's inequality together with (5.1) yields

$$1 \leq \frac{2C}{k} \frac{N-1}{N} \sum_{j=1}^N \left\| \tilde{\Phi}^{-1} \left( \Phi_* \left( \frac{|u|}{k} \right) \right) \right\|_{\tilde{\Phi}, \Omega_{\epsilon,i}} \left\| \frac{\partial u}{\partial x_j} \right\|_{\Phi, \Omega_{\epsilon,i}} + C \int_{\Omega_{\epsilon,i}} \left| v \left( \frac{|u|}{k} \right) \right| dx. \quad (5.2)$$

Now, a direct computation leads to

$$\int_{\Omega_{\epsilon,i}} \left| v \left( \frac{|u|}{k} \right) \right| dx \leq \frac{2}{k} \frac{N-1}{N} \left\| \tilde{\Phi}^{-1} \left( \Phi_* \left( \frac{|u|}{k} \right) \right) \right\|_{\tilde{\Phi}, \Omega_{\epsilon,i}} \|u\|_{\Phi, \tilde{\Omega}_{\epsilon,i}}.$$

Since

$$\left\| \tilde{\Phi}^{-1} \left( \Phi_* \left( \frac{|u|}{k} \right) \right) \right\|_{\tilde{\Phi}, \Omega_{\epsilon,i}} \leq 1,$$

we get,

$$\int_{\Omega_{\epsilon,i}} \left| v \left( \frac{|u|}{k} \right) \right| dx \leq \frac{2}{k} \frac{N-1}{N} \|u\|_{\Phi, \tilde{\Omega}_{\epsilon,i}}. \quad (5.3)$$

From (5.2)-(5.3),

$$1 \leq \frac{2C}{k} \frac{N-1}{N} \|\nabla u\|_{\Phi, \tilde{\Omega}_{\epsilon,i}} + \frac{2}{k} \frac{N-1}{N} \|u\|_{\Phi, \tilde{\Omega}_{\epsilon,i}}.$$

Hence, there exists  $M_* > 0$ , independent of  $\epsilon$  such that

$$\|u\|_{\Phi_*, \Omega_{\epsilon, i}} \leq M_* \|u\|_{\tilde{X}_{\epsilon, i}} \quad \text{for all } u \in \tilde{X}_{\epsilon, i} \cap L^\infty(\Omega_{\epsilon, i}),$$

obtaining the desired result.  $\blacksquare$

As a byproduct of the above proof, we have the following corollary

**Corollary 5.1** *Let  $(y_{n, i})$  the sequence obtained in (4.6). There is  $C > 0$ , which is independent of  $\rho$  and  $n \in \mathbb{N}$ , such that*

$$\int_{\mathbb{R}^N \setminus \cup_{j=1}^p B_\rho(y_{n, i})} \Phi_*(|v|) dx \leq C \|v\|_{W^{1, \Phi}(\mathbb{R}^N \setminus \cup_{j=1}^p B_\rho(y_{n, i}))},$$

for all  $v \in W^{1, \Phi}(\mathbb{R}^N \setminus \cup_{j=1}^p B_\rho(y_{n, i}))$ .

**Proof.** The corollary follows by repeating the same steps used in the proof Proposition 5.1. The main point that we would like to point out is the fact that the constant associated with the embedding

$$W^{1, 1}(\mathbb{R}^N \setminus \cup_{j=1}^p B_\rho(y_{n, i})) \hookrightarrow L^{\frac{N}{N-1}}(\mathbb{R}^N \setminus \cup_{j=1}^p B_\rho(y_{n, i}))$$

is also independent of  $\rho$  and  $n \in \mathbb{N}$ , because  $\Theta_{\rho, n, i} = \mathbb{R}^N \setminus \cup_{j=1}^p B_\rho(y_{n, i})$  verifies the uniform cone condition for all  $\rho > 0$  and  $n \in \mathbb{N}$ .  $\blacksquare$

The next result is also well known for Sobolev spaces, however for Orlicz-Sobolev spaces we do not know any reference. Here, we adapt some arguments found in [3].

**Proposition 5.2** *Let  $\varrho > 0$  and  $\epsilon_n \in (0, +\infty)$  with  $\epsilon_n \rightarrow 0$ . Let  $v_{n, i} \in \tilde{X}_{\epsilon_n, i}$  be a sequence and a constant  $C_0 > 0$  such that*

$$\|v_{n, i}\|_{\tilde{X}_{\epsilon_n, i}} \leq C_0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_\varrho(y) \cap \Omega_{\epsilon_n, i}} \Phi(|v_{n, i}|) dx = 0.$$

Then,

$$\lim_{n \rightarrow +\infty} \int_{\Omega_{\epsilon_n, i}} B(|v_{n, i}|) dx = 0,$$

for any  $N$ -function  $B$  verifying  $\Delta_2$ -condition,

$$\lim_{t \rightarrow 0} \frac{B(t)}{\Phi(t)} = 0 \quad \text{and} \quad \lim_{|t| \rightarrow +\infty} \frac{B(t)}{\Phi_*(t)} = 0.$$

**Proof.** Firstly, note that given  $\eta > 0$  there exists  $\kappa > 0$  such that

$$B(|v_{n,i}|) \leq \eta \Phi_*(|v_{n,i}|), \text{ for } |v_{n,i}| \geq \kappa.$$

As  $(\|v_{n,i}\|_{\tilde{X}_{\epsilon_n,i}})$  is bounded in  $\mathbb{R}$ , we have

$$\int_{\Omega_{\epsilon_n,i}} B(|v_{n,i}|) dx \leq \eta C + \int_{\Omega_{\epsilon_n,i} \cap [|v_{n,i}| \leq \kappa]} B(|v_{n,i}|) dx$$

which implies

$$\limsup_{n \rightarrow +\infty} \int_{\Omega_{\epsilon_n,i}} B(|v_{n,i}|) dx \leq \eta C + \limsup_{n \rightarrow +\infty} \int_{\Omega_{\epsilon_n,i} \cap [|v_{n,i}| \leq \kappa]} B(|v_{n,i}|) dx. \quad (5.4)$$

We will show that

$$\limsup_{n \rightarrow +\infty} \int_{\Omega_{\epsilon_n,i} \cap [|v_{n,i}| \leq \kappa]} B(|v_{n,i}|) dx = 0. \quad (5.5)$$

For this purpose, we consider for each  $\zeta > 0$  enough small, the function  $\chi_\zeta \in C_0^1(\mathbb{R})$  given by

$$\chi_\zeta(s) = \begin{cases} 1, & \text{if } |s| \leq \kappa - \zeta, \\ a_1(s), & \text{if } -(\kappa + \zeta) \leq s \leq -(\kappa - \zeta), \\ a_2(s), & \text{if } \kappa - \zeta \leq s \leq \kappa + \zeta, \\ 0, & \text{if } |s| \geq \kappa + \zeta, \end{cases}$$

where  $a_1, a_2 \in C^1(\mathbb{R}; [0, 1])$ ,  $a_1$  is nondecreasing and  $a_2$  is nonincreasing. Next, let us define the auxiliary function

$$u_{n,i}(x) = \chi_\zeta(|v_{n,i}(x)|) v_{n,i}(x).$$

Notice that

$$\int_{\Omega_{\epsilon_n,i}} B(|u_{n,i}|) dx \geq \int_{\Omega_{\epsilon_n,i} \cap [|v_{n,i}| \leq \kappa - \zeta]} B(|v_{n,i}|) dx. \quad (5.6)$$

Thereby, (5.5) follows by showing the limit below

$$\limsup_{n \rightarrow +\infty} \int_{\Omega_{\epsilon_n,i}} B(|u_{n,i}|) dx = 0. \quad (5.7)$$

In fact, gathering the above limit with (5.6), we derive that

$$\limsup_{n \rightarrow +\infty} \int_{\Omega_{\epsilon_n,i} \cap [|v_{n,i}| \leq \kappa - \zeta]} B(|v_{n,i}|) dx = 0.$$

Since

$$\int_{\Omega_{\epsilon_n, i} \cap [\kappa - \zeta \leq |v_{n, i}| \leq \kappa]} B(|v_{n, i}|) dx = o_n(1)$$

and

$$\begin{aligned} \int_{\Omega_{\epsilon_n, i} \cap [|v_{n, i}| \leq \kappa]} B(|v_{n, i}|) dx &= \int_{\Omega_{\epsilon_n, i} \cap [\kappa - \zeta \leq |v_{n, i}| \leq \kappa]} B(|v_{n, i}|) dx + \\ &\quad \int_{\Omega_{\epsilon_n, i} \cap [|v_{n, i}| \leq \kappa - \zeta]} B(|v_{n, i}|) dx, \end{aligned}$$

we deduce that

$$\limsup_{n \rightarrow +\infty} \int_{\Omega_{\epsilon_n, i} \cap [|v_{n, i}| \leq \kappa]} B(|v_{n, i}|) dx = 0,$$

showing (5.5). Now, by (5.4) and (5.5),

$$\limsup_{n \rightarrow +\infty} \int_{\Omega_{\epsilon_n, i}} B(|v_{n, i}|) dx \leq \eta C.$$

By using that  $\eta$  is arbitrary, it follows that

$$\limsup_{n \rightarrow +\infty} \int_{\Omega_{\epsilon_n, i}} B(|v_{n, i}|) dx = 0,$$

proving the proposition. Now, we observe that (5.7) follows by repeating the same approach explored in [3, Theorem 3.1]. ■

## References

- [1] A. Adams and J.F. Fournier, Sobolev Spaces, 2nd ed., Academic Press (2003).
- [2] C.O. Alves, Existence of multi-peak of solutions for a class of quasilinear problems in  $\mathbb{R}^N$ , Topol. Methods Nonlinear Anal. 38, (2011) 307-332.
- [3] C.O. Alves, G.M. Figueiredo and J.A. Santos, Strauss and Lions type results for a class of Orlicz-Sobolev spaces and applications, Topol. Methods Nonlinear Anal. 44, no. 2, (2014) 435-456.



- [4] C.O. Alves and A.R. da Silva, Multiplicity and concentration of positive solutions for a class of quasilinear problems through Orlicz-Sobolev space, arXiv:1506.01669v1.
- [5] C.O. Alves and A.R. da Silva, Multiplicity and concentration behavior of solutions for a quasilinear problem involving N-functions via penalization method, *Electron. J. Differential Equations* 2016 (2016), No. 158, 1- 24.
- [6] A. Ambrosetti, M. Badiale and S. Cingolani, Semiclassical states of nonlinear Schrödinger equations, *Arch. Rational Mech. Anal.*, 140 (1997), 285-300.
- [7] A. Azzollini, P. d'Avenia and A. Pomponio, Quasilinear elliptic equations in  $\mathbb{R}^N$  via variational methods and Orlicz-Sobolev embeddings, *Calc. Var. Partial Differential Equations* 49, (2014) 197-213.
- [8] G. Bonanno, G.M. Bisci and V. Radulescu, Quasilinear elliptic non-homogeneous dirichlet problems through Orlicz-Sobolev spaces, *Nonlinear Anal.* 75, (2012) 4441-4456.
- [9] G. Bonanno, G.M. Bisci and V. Radulescu, Existence and multiplicity of solutions for a quasilinear nonhomogeneous problems: An Orlicz-Sobolev space setting *J. Math. Anal. Appl.* 330, (2007) 416-432.
- [10] H. Brezis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* 88, no. 3, (1983)
- [11] S. Cingolani and M. Lazzo, Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations, *Topol. Methods. Nonl. Analysis* 10 (1997), 1-13.
- [12] M. del Pino and P.L. Felmer, Local mountain pass for semilinear elliptic problems in unbounded domains, *Calc. Var. Partial Differential Equations* 4, no. 2, (1996) 121-137.
- [13] M. del Pino and P.L. Felmer, Multi-peak bound states for nonlinear Schrödinger equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 15, (1998) 127-149.
- [14] E. DiBenedetto,  $C^{1,\gamma}$  local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.* 7, no. 8, (1985) 827-850.

- [15] T.K. Donaldson and N.S. Trudinger, Orlicz-Sobolev spaces and imbedding theorems, *J. Funct. Anal.* 8, no. 1, (1971) 52-75.
- [16] A. Floer and A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, *J. Funct. Anal.* 69, no. 3, (1986) 397-408.
- [17] N. Fukagai, M. Ito and K. Narukawa, Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev nonlinearity on  $\mathbb{R}^N$ , *Funkcial. Ekvac.* 49, no. 2, (2006) 235-267.
- [18] N. Fukagai, M. Ito and K. Narukawa, Quasilinear elliptic equations with slowly growing principal part and critical Orlicz-Sobolev nonlinear term, *Proc. Roy. Soc. Edinburgh Sect. A* 139, no. 1, (2009) 73-106.
- [19] N. Fukagai and K. Narukawa, On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problems, *Ann. Mat. Pura Appl.* 186, no. 3, (2007) 539-564.
- [20] N. Fusco and C. Sbordone, Some remarks on the regularity of minima of anisotropic integrals, *Comm. Partial Differential Equations* 18, no. 1-2, (1993) 153-167.
- [21] A. Giacomini and M. Squassina, Multi-peak solutions for a class of degenerate elliptic equations, *Asymptotic Anal.* 36, (2003) 115-147.
- [22] C. Gui, Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method, *Comm. Partial Differential Equations* 21, no. 5-6, (1996) 787-820.
- [23] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and quasilinear elliptic equations*, Acad. Press (1968).
- [24] V.K. Le, D. Motreanu and V.V. Motreanu, On a non-smooth eigenvalue problem in Orlicz-Sobolev spaces., *Appl. Anal.* 89, no. 2, (2010) 229-242.
- [25] M. Mihailescu, V. Radulescu and D. Repovš, On a non-homogeneous eigenvalue problem involving a potential: an Orlicz-Sobolev space setting, *J. Math. Pures Appl.* 93, (2010) 132-148.
- [26] M. Mihailescu and V. Radulescu, Nonhomogeneous Neumann problems in Orlicz-Sobolev spaces. *C.R. Acad. Sci. Paris, Ser. I* 346, (2008) 401-406.

- [27] M. Mihailescu and V. Radulescu, Existence and multiplicity of solutions for a quasilinear non-homogeneous problems: An Orlicz-Sobolev space setting, *J. Math. Anal. Appl.* 330, (2007) 416-432.
- [28] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Vol. 1034, Springer, Berlin, 1983.
- [29] Y.G. Oh, On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential, *Comm. Math. Phys.* 131, no. 2, (1990) 223-253.
- [30] M.N. Rao and Z.D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York (1985).
- [31] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, *Z. Angew. Math. Phys.* 43, no. 2, (1992) 270-291.
- [32] J.A. Santos, Multiplicity of solutions for quasilinear equations involving critical Orlicz-Sobolev nonlinear terms, *Electron. J. Differential Equations* 2013, no. 249, (2013) 1-13.
- [33] J.A. Santos and S.H.M. Soares, Radial solutions of quasilinear equations in Orlicz-Sobolev type spaces, *J. Math. Anal. Appl.* 428, (2015) 1035-1053.
- [34] X. Wang, On concentration of positive bound states of nonlinear Schrödinger equations, *Comm. Math. Phys.*, 153, no. 2 (1993) 229-244.
- [35] Zhen-hui Zhang and Hao-yuan Xu, Existence of multi-peak solutions for  $p$ -Laplace problems in  $\mathbb{R}^N$ , *Acta Mathematicae Applicatae Sinica* 31, no. 4 (2015) 1061-1072.